Calculus 140, section 3.3 Derivatives of Combinations of Functions

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What we have so far:

The (first) derivative of a function *f* is given by $f'(x) = \lim\limits_{t \to x} \dfrac{f(t) - f(x)}{t - x} = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$. [section 3.1]

The derivative of *f* has several notations: f', $f'(x)$, $\dfrac{dy}{dx}$, $\dfrac{d}{dx}[f(x)]$.

(The latter two are Leibniz notation.)

If *f* is differentiable at *a*, then *f* is continuous at *a*, that is $\lim\limits_{x \to a} f(x) = f(a)$. [Thm 3.2]

If *f* is differentiable at each number in its domain, then *f* is a **differentiable function**. [Def 3.3]

Given a function $f(x) = x^n$ where *n* is a positive integer, $f'(x) = n\,x^{n-1}$. [section 3.2]

$$\frac{d}{dx}[\sin x] = \cos x, \quad \frac{d}{dx}[\cos x] = -\sin x, \quad \frac{d}{dx}\left[e^x\right] = e^x \text{ for all } -\infty < x < \infty.$$

Section 3.3 takes us a little further, and gives us some more rules (shortcuts) for finding derivatives.

The derivative formulas above from section 3.2 applied only to single-term functions, not to sums, differences, products or quotients.

What happens to a derivative if we add two functions together? Given $p(x) = f(x) + g(x)$, does

$$p'(x) = f'(x) + g'(x)?$$

By the sum rule for limits, $\lim\limits_{h \to 0} \dfrac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} = \lim\limits_{h \to 0} \dfrac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h}$

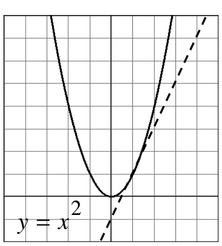
$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

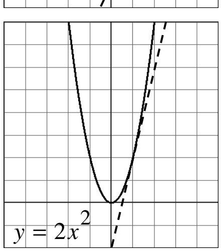
Theorem 3.4 [**Sum Rule**]: If *f* and *g* are differentiable, then

$$(f+g)'(a) = f'(a) + g'(a) \text{ and } (f+g)'(x) = f'(x) + g'(x).$$

Or, using Leibniz notation, given variables *u* and *v* which are dependent on *x*, $\dfrac{d}{dx}(u+v) = \dfrac{du}{dx} + \dfrac{dv}{dx}$.

Given a function $y = c * f(x)$, where *c* is a constant coefficient of a variable function, what would be its first derivative? Think in terms of what you know about transformations and what you've learned about slope of the tangent line. The constant will either stretch the graph (when $|c| > 1$) or shrink the graph (when $|c| < 1$). What effect will this stretch/shrink have on the slope of the tangent line? Will it stretch/shrink at the same rate as the curve? Consider the quadratics pictured to the left. For $f(x) = x^2$, $f(1) = 1$ and the slope of the tangent line = 2. For $f(x) = 2x^2$, $f(1) = 2$ and the slope of the tangent line = 4. We might begin to suspect that as a function undergoes a stretch/shrink, the tangent line stretches/shrinks at the same rate. Indeed, we already know, from the properties of limits, that





$$\lim_{h \to 0} \frac{c f(x+h) - c f(x)}{h} = \lim_{h \to 0} \frac{c[f(x+h) - f(x)]}{h} = c * \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

That is, given a differentiable function *f* and a constant *c*,

$\dfrac{d}{dx}[c * f(x)] = c * f'(x)$ [Theorem 3.5: **constant-multiple rule**].

Note that the constant-multiple rule and sum rule work together to give us the derivative of a subtraction, since

$$f(x)-g(x)=f(x)+(-1*g(x)),\text{ so that }(f-g)'(x)=f'(x)-g'(x).$$

Example A: Find $\frac{d}{dx}\left(6x^5-2\cos x-4e^x\right)\Big|_{x=\pi/3}$. *answer*: $\frac{10\pi^4}{27}+\sqrt{3}-4e^{\pi/3}$

Caution, Be careful, Warning, Warning! Danger, Will Robinson! There is no similar easy process for the derivative of a product, nor is there a similar easy process for the derivative of a quotient. The formulas are quite a bit more complicated.

Theorem 3.6 [**product rule**]: If f and g are differentiable, then

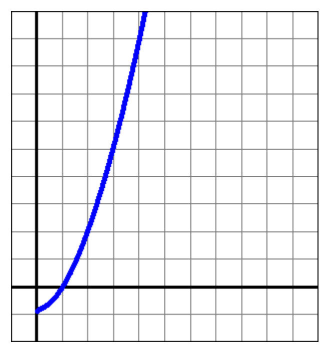
$$(f*g)'(x)=f(x)*g'(x)+g(x)*f'(x)\text{ or }\frac{d}{dx}[u*v]=u*\frac{dv}{dx}+v*\frac{du}{dx}.$$

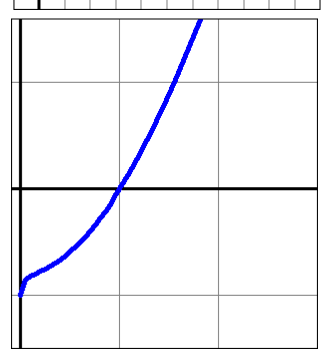
The text does a derivation in the middle of the chapter. I'll leave it to you to look over that work.

Side note: I am aware that my version looks slightly different than the product rule in your text. I have made use of the fact that multiplication and addition are both commutative and associative.

Example B: Given $y=(2x+1)(\sqrt{x}-1)$ solve $\frac{dy}{dx}=0$.

answer: no solution





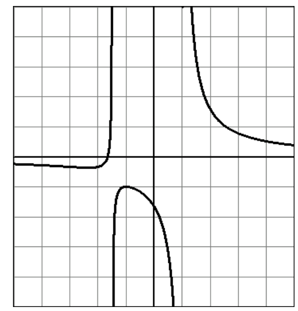
Theorem 3.7 [**quotient rule**]: If f and g are differentiable, and $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x) * f'(x) - f(x) * g'(x)}{[g(x)]^2} \quad \text{or} \quad \frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v * \frac{du}{dx} - u * \frac{dv}{dx}}{v^2}.$$

The text does a derivation in the middle of the chapter. I'll leave it to you to look over that work.

Side note: Unlike the product rule, order is vital. Subtraction and division are not commutative.

Example C: Find the two x-values where $f(x) = \dfrac{3x+5}{2x^2+x-3}$ has a horizontal tangent. *answers*: $x = -\dfrac{7}{3}, -1$



Check out the text's Example 11, which uses the quotient rule to differentiate $y = \tan x = \dfrac{\sin x}{\cos x}$, and shows that $\dfrac{d}{dx}[\tan x] = \sec^2 x$ and $\dfrac{d}{dx}[\cot x] = -\csc^2 x$. These are worth memorizing.

Also check out the text's Example 12, which shows that $\dfrac{d}{dx}[\sec x] = \sec x \tan x$ and $\dfrac{d}{dx}[\csc x] = -\csc x \cot x$. These are going to be useful, too.

Example D: Find $\dfrac{d}{dx}\left[x^{-a}\right]$, where a is a positive integer. *answer*: $-a\,x^{-a-1}$

This Example is sufficient justification to expand the power rule for derivatives from "only positive integer exponents" to "any nonzero integer exponent".
Given a function $f(x) = x^n$ where n is a nonzero integer, $f'(x) = n\,x^{n-1}$.

Examples E: Given the functions $y = \dfrac{x^5}{5}$, $y = \dfrac{5}{x^5}$ and $y = \dfrac{1}{5x^5}$ find the three first derivatives.

answers: $x^4, -\dfrac{25}{x^6}, -\dfrac{1}{x^6}$