Calculus 140, section 3.4 The Chain Rule

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What we have so far:

Given a function $f(x) = x^n$ where n is a non-zero integer, $f'(x) = n\,x^{n-1}$ [section 3.2, 3.3]

$$\frac{d}{dx}[\sin x] = \cos x, \quad \frac{d}{dx}[\cos x] = -\sin x, \quad \frac{d}{dx}\left[e^x\right] = e^x \text{ for all } -\infty < x < \infty \text{ [section 3.2]}$$

$$(f+g)'(x) = f'(x) + g'(x) \text{ [Thm 3.4]}, \quad \frac{d}{dx}[c * f(x)] = c * f'(x) \text{ [Thm 3.5]}$$

$$(f * g)'(x) = f(x) * g'(x) + g(x) * f'(x) \text{ [Thm 3.6]}, \quad \left(\frac{f}{g}\right)'(x) = \frac{g(x) * f'(x) - f(x) * g'(x)}{[g(x)]^2} \text{ [Thm 3.7]}$$

Composition of functions is taking one function's formula and inserting it into another's. Vocabulary and notation varies: "g composition f" = "g of f" = $g \circ f = (g \circ f)(x) = g[f(x)]$.

Example A: Given $f(x) = 3x + 1$ and $g(x) = x^2 - x$, find the algebraic rules for $(g \circ f)(x)$ and $(g \circ f)'(x)$.
answers: $9x^2 + 3x$; $18x + 3$

Theorem 3.8 [Chain Rule] If a function f is differentiable at x, and a function g is differentiable at $f(x)$, then

$$\frac{d}{dx} g[f(x)] = g'[f(x)] * f'(x).$$

Your text has a short justification at the beginning of the section, and a full proof in the Appendix.

Example B: Given $p(x) = \sqrt{4x^3 - 3}$ and $q(x) = 2x^2 - x$, find $(p[q(x)])'$. *answer*: $\dfrac{6\left(2x^2 - x\right)^2 (4x - 1)}{\sqrt{4\left(2x^2 - x\right)^3 - 3}}$

One way to think of the chain rule is "the derivative of the outside applied to the inside, times the derivative of the inside". Possibly the most important task in using the chain rule is correctly identifying the "outside and "inside" functions, i.e. which one is "*g* " and which one is "*f*".

Example C: Given $h(x) = \left(x + \sqrt{x}\right)^3$ find $h'(x)$. *answer*: $3\left(x+\sqrt{x}\right)^2\left(\frac{2\sqrt{x}+1}{2\sqrt{x}}\right)$

Example D: Suppose that *y* is a differentiable function of *x*. Express the derivative of $y^4 \cos 2x$ in terms of *x*, *y*, and $\dfrac{dy}{dx}$.

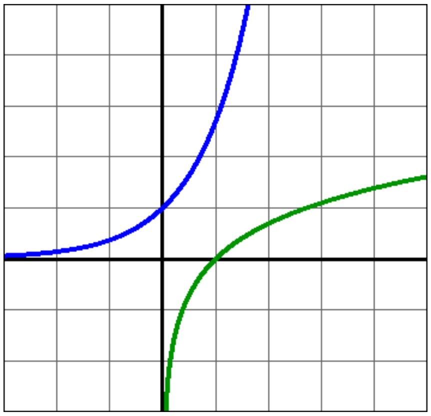
We have an alternate way of writing the chain rule using Leibniz notation: for $y = f(u)$ and $u = g(x)$,

$$\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}.$$

Example E: An environmental study determined that the level of carbon monoxide (in parts per million) in the air surrounding a small city was a function of the number of people living there (*p* = population in thousands): $C(p) = \sqrt{0.5p^2 + 17}$. Population is, in turn, a function of time (in years): $p(t) = 3.1 + 0.1t^2$.

a) Explain what $\dfrac{dC}{dp}$, $\dfrac{dp}{dt}$, and $\dfrac{dC}{dt}$ represent. b) Use the chain rule to find $\left.\dfrac{dC}{dt}\right|_{t=3}$. *answer* b) 0.24

Examples F: Find the derivatives of the functions $y = e^{cx}$ and $y = \pi^x$. *answers*: $c\,e^{cx}$, $(\ln \pi) * \pi^x$



The *natural logarithm* function, $y = \ln(x)$, is the inverse of the natural exponential function, $y = e^x$. By finding the first derivative, we can determine that the slope of $y = e^x$ is 1 at the point (0, 1), i.e. $\left.\frac{d}{dx}\left(e^x\right)\right|_{x=0} = 1$. By symmetry, the slope of $y = \ln(x)$ should also be 1 at the point (1, 0), i.e. $\left.\frac{d}{dx}(\ln x)\right|_{x=1} = 1$. Also recall that $y = \ln(x)$ is increasing over its entire domain The formula we use for the derivative of ln(x) must meet these conditions.

Finding a derivative formula for ln(x) is actually quite simple. First note that since $e^{\ln x} = x$, then $\frac{d}{dx}\left(e^{\ln x}\right) = \frac{d}{dx}(x) = 1$. By the chain rule, $\frac{d}{dx}\left(e^{\ln x}\right) = e^{\ln x} * \frac{d}{dx}(\ln x) = x * \frac{d}{dx}(\ln x) = 1 \Rightarrow \frac{d}{dx}(\ln x) = \frac{1}{x}$.

Note that $\left.\frac{d}{dx}(\ln x)\right|_{x=1} = \left.\frac{1}{x}\right|_{x=1} = 1$. Also $\frac{d}{dx}(\ln x) = \frac{1}{x} > 0$ for all x in the domain of ln(x).
In other words, the "necessary conditions" listed above have been met.

Example G: Find the derivative of functions $y = x^r$, where r is any Real number except 0. *answer*: $r\,x^{r-1}$

Example H. Given $f(x) = \ln\left(\sqrt{x^5} + 1\right)$, find $f'(x)$. *answer*: $\frac{5\sqrt{x^3}}{2\left(\sqrt{x^5}+1\right)}$