Calculus 140, section 3.5 Higher Order Derivatives

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Given a function $f(x) = x^r$ where r is a non-zero real number, $f'(x) = r\,x^{r-1}$ [section 3.2, 3.3, 3.4]

$$\frac{d}{dx}[\sin x] = \cos x, \quad \frac{d}{dx}[\cos x] = -\sin x, \quad \frac{d}{dx}\left[e^{x}\right] = e^{x}, \quad \frac{d}{dx}[\ln x] = \frac{1}{x} \quad \text{[section 3.2, 3.4]}$$

$$(f+g)'(x) = f'(x) + g'(x) \text{ [Thm 3.4]}, \qquad \frac{d}{dx}[c * f(x)] = c * f'(x) \text{ [Thm 3.5]}$$

$$(f * g)'(x) = f(x) * g'(x) + g(x) * f'(x) \text{ [Thm 3.6]}, \qquad \left(\frac{f}{g}\right)'(x) = \frac{g(x) * f'(x) - f(x) * g'(x)}{[g(x)]^2} \text{ [Thm 3.7]}$$

$$\frac{d}{dx} g[f(x)] = g'[f(x)] * f'(x) \text{ [Thm 3.8]}$$

Recall, however, that the first derivative is itself a function, which has its own domain and its own graph. Since it is a function, it also has its own derivative. Given a function f, we can calculate the first derivative f' or $\frac{dy}{dx}$.

We can then calculate the derivative of f', i.e. the second derivative of f, symbolically $f''(x) = \frac{d^2 y}{dx^2}$.

<u>Important note</u>: Just like $\frac{dy}{dx}$ is *not* a fraction, but is a notation for the first derivative, $\frac{d^2 y}{dx^2}$ is also not a fraction but a notation. *There is no multiplication involved!* Rather, you need to interpret it this way:

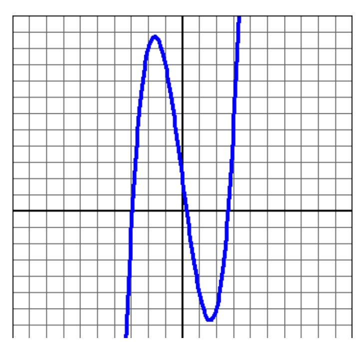
$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$ which means "the derivative of $\frac{dy}{dx}$", the derivative of a derivative.

In a similar fashion, we can find higher-order derivatives.

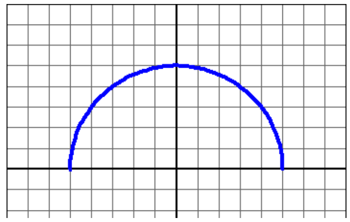
$$\frac{d}{dx}[f(x)] = f'(x) = \frac{dy}{dx}, \quad \frac{d}{dx}[f'(x)] = f''(x) = \frac{d^2 y}{dx^2}, \quad \frac{d}{dx}[f''(x)] = f^{(3)}(x) = \frac{d^3 y}{dx^3}, \quad \frac{d}{dx}\left[f^{(3)}\right] = f^{(4)}(x) = \frac{d^4 y}{dx^4}$$

$$\frac{d}{dx}\left[f^{(n-1)}\right] = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

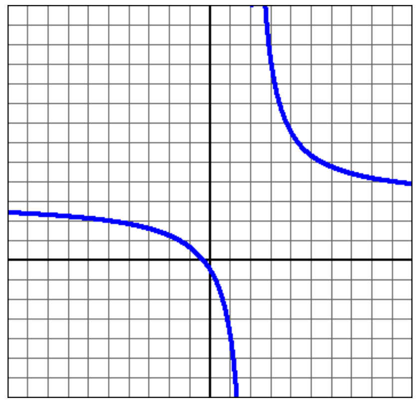
Example A: Given $f(x) = x^3 - 8x + 2$, find all higher derivatives of f.



Example B: Given $f(x)=\sqrt{25-x^2}$, find the second derivative. *answer*: $\frac{-25}{\left(25-x^2\right)^{3/2}}$



Example C: Given $f(x)=\dfrac{3x+1}{x-2}$, find $f(x)=\dfrac{d^3y}{dx^3}$. *answer*: $\frac{-42}{(x-2)^4}$



In Examples B and C, we needed the Quotient Rule. Your text, in Example 6, uses the Product Rule, to find the first three derivatives of $y = x \sin x$.

The text also, in Example 5, demonstrates that for functions $f(x)=e^{cx}$, the n^{th} derivative is $f^{(n)}(x)=c^n e^{cx}$.

Example D: Given $f(x)=\ln x$, find a formula for the n^{th} derivative of f for $n \ge 1$. *answer*: $(-1)^{n-1}\left[(n-1)!\right]x^{-n}$

We have already, in earlier sections, determined that velocity is the first derivative of a position function.
But velocity is not always constant. Rather, it changes. Sometimes we go slower; sometimes we speed up.
The rate of change of velocity is called acceleration.
The derivative of velocity is acceleration.
The derivative of [the derivative of position] is acceleration.
The second derivative of a position function is acceleration.
(Read through the text's explanation and example 7.)

Example E: The three graphs below (in no particular order) are graphs of $h(t)$, the height of a toy helicopter above the ground, $v(t)$ [the velocity of the helicopter], and $a(t)$ [the acceleration of the helicopter]. Use your knowledge of first and second derivatives to determine which graph is of which function. Justify your answer.