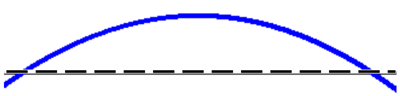
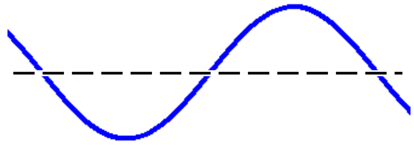
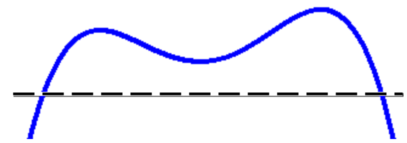
Calculus 140, section 4.2 The Mean Value Theorem

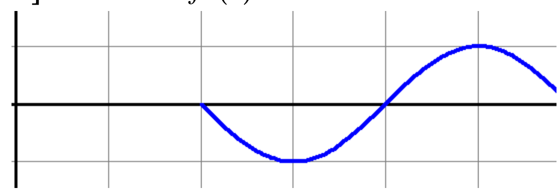
notes by Tim Pilachowski

We begin with Rolle's Theorem [Theorem 4.4] (named for Michel Rolle): "Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there is a number c in (a, b) such that $f'(c) = 0$."

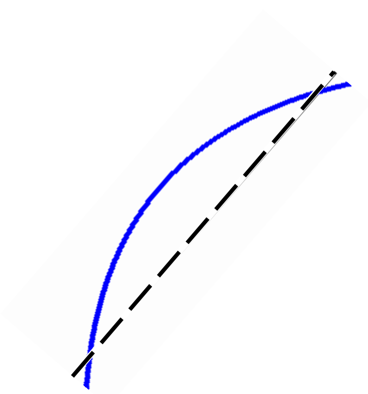
If f is a constant function, then $f'(x) = 0$ for all values in (a, b). If f is not a constant function, then by Theorem 4.2 (Maximum-Minimum Theorem) the maximum and minimum values are distinct. Since $f(a) = f(b)$, either the maximum or minimum (or possibly both) must occur at an interior point c on (a, b). Because, by hypotheses, f is differentiable on (a, b), and therefore at c, then by Theorem 4.3 $f'(c) = 0$.

Example A: Given $f(x) = \sin x$, find all numbers c in the interval $[\pi, 3\pi]$ for which $f'(c) = 0$.



Rolle's Theorem depends on the condition that $f(a) = f(b)$. If this condition is not met, then we may not have a number c in (a, b) such that $f'(c) = 0$.



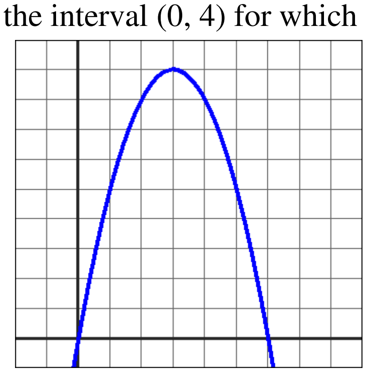
Intuitively, however, when f is continuous on $[a, b]$ and $f(a) \neq f(b)$, it seems as though it must be true that there is a value c on (a, b) where the tangent at c is parallel to the secant line connecting $(a, f(a))$ to $(b, f(b))$.

Theorem 4.5 [Mean Value Theorem]: "Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}."$$

In the text there is a five-line proof which uses an intermediate function g and Rolle's Theorem.

Example B: Consider $f(x) = x(6 - x)$ on the interval $[0, 4]$. Find all numbers c in the interval $(0, 4)$ for which the line tangent to the graph is parallel to the line joining $(0, 0)$ and $(4, 8)$.



The equation in Example B was solved using basic algebra. One of your text practice exercises has you use the Newton-Raphson method to estimate a value for c for which the line tangent to the graph is parallel to the secant line joining two points.

Example C: I leave my house, and travel the 22 miles to UMCP five days a week when school is in session.
a) If the trip takes me 30 minutes, must I have exceeded the 55 mph speed limit on the BW parkway?
b) What travel time would prove that I must have exceeded that 55 mph speed limit?

Text Exercise 15 asserts that if $|f'(x)| \leq M$ on [a, b], then $f(a) - M(b-a) \leq f(b) \leq f(a) + M(b-a)$.

Example D [text exercise #18]: Use this result from Exercise 15 to determine lower and upper bounds for $33^{1/5}$.