Calculus 140, section 4.3 Consequences of the Mean Value Theorem
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In mathematics we often go backwards. Subtraction is the "backwards" of addition: $2+3=5 \rightarrow 5-3=2$. Division is the "backwards" of multiplication: $4*5=20 \rightarrow 20\div5=4$. Solving is the "backwards" of calculation: $3(2+5)=30 \rightarrow 3(x+5)=30$.

An antiderivative is the "backwards" of finding the first derivative. In the real world, we may know something about the rate of change from experiment or observation, and want to work our way backwards to find the equation that describes a phenomenon.

Example A: Given a function $f(x)=5x^4$ find a function $F(x)$ such that $F'(x)=f(x)$. *answer*: $F(x)=x^5+C$

Example A extended: Given a function $f(x)=5x^4$ find a function $F(x)$ such that $F'(x)=f(x)$ and $F(1)=8$. *answer*: $F(x)=x^5+7$

Theorem 4.6: "a. Let f be continuous on an interval I. If $f'(x)$ exists and equals 0 for each interior point x of I, then f is constant on I.

      b. Let f and g be continuous on an interval I. If $f'(x)$ and $g'(x)$ exist and are equal for each interior point x of I, then $f-g$ is constant on I. In other words, there is a constant C such that $f(x)=g(x)+C$ for all x in I."

The proof relies on the mean Value Theorem.

Example B: Find all antiderivatives of $f(x) = x^{-1}$. *answer:* $F(x) = \ln|x| + C$

Example C: Find the function satisfying the conditions $f'(x) = e^{-2x}$ and $f(0) = 3$.
answer: $f(x) = -\frac{1}{2}e^{-2x} + \frac{7}{2}$

Example D: Find the function satisfying the conditions $f''(x) = \cos x$, $f'\left(\frac{\pi}{2}\right) = 3$ and $f(0) = 1$.
answer: $f(x) = -\cos x + 2x + 2$

Theorem 4.7: "Let f be continuous on an interval I and differentiable at each interior point of I.
    a. If $f'(x) > 0$ at each interior point of I, then f is increasing on I. Moreover, f is increasing on I if $f'(x) > 0$
except for a finite number of points x in I.
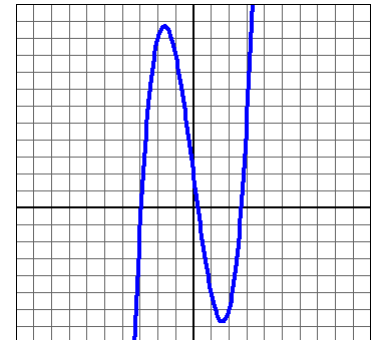   b. If $f'(x) < 0$ at each interior point of I, then f is decreasing on I. Moreover, f is decreasing on I if
$f'(x) < 0$ except for a finite number of points x in I.

The proof, done in the text, relies on the MVT.

Example E: Find intervals on which the function $f(x) = x^3 - 8x + 2$ is increasing and those on which it is decreasing.
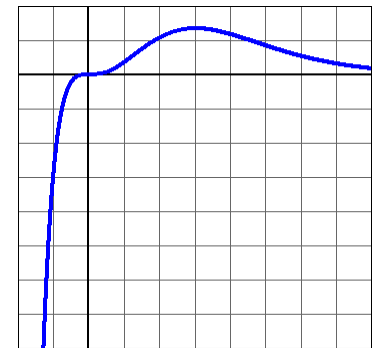
first derivative:

critical numbers:

interval(s) increasing:

interval(s) decreasing:

Example F: Consider the function $f(x) = \dfrac{x^3}{e^x}$.

first derivative:

critical numbers:

interval	$f'(x)$	f

interval(s) increasing:

interval(s) decreasing:

Special note about the graph of f in an interval surrounding the point (0, 0):

Example G: Find intervals on which the function $f(x) = \dfrac{1}{x}$ is increasing and those on which it is decreasing.

Example G revisited: Can we say that the function $f(x) = \dfrac{1}{x}$ is decreasing over its entire domain?