Calculus 140, section 5.2 The Definite Integral

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Spoiler alert: Much of this Lecture is going to look a lot like the one for section 5.1.

In Example B from Lecture 5.1, we noted that increasing the number of subintervals in our partition brought us closer to the true value for the area under the curve $y = 2\sqrt{x}$. It is reasonable to suppose that, in general, for any function, increasing the number of subintervals will provide an increasingly better approximation to the area under the curve.

Now, in section 5.2 we look at the limit of a Riemann sum as the number of subintervals in the partition *n* approaches ∞, or alternately, as the lengths of the subintervals approaches 0.

Notation: If we use $\|P\|$, the **norm** of a partition *P*, to mean the length of the largest subinterval associated with *P*, then we're looking at $\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(t_k)\,\Delta x_k$.

Theorem 5.3: "For any function *f* that is continuous on [*a*, *b*], there is a number *I* with the following property: For any $\varepsilon > 0$, there is a number $\delta > 0$ such that if $P = \{x_0, x_1, \ldots, x_n\}$ is any partition of [*a*, *b*] each of whose subintervals has length less than δ, and if $x_{k-1} \le t_k \le x_k$ for each *k* between 1 and *n*, then the associated Riemann sum $\sum_{k=1}^{n} f(t_k)\,\Delta x_k$ satisfies $\left| I - \sum_{k=1}^{n} f(t_k)\,\Delta x_k \right| < \varepsilon$."

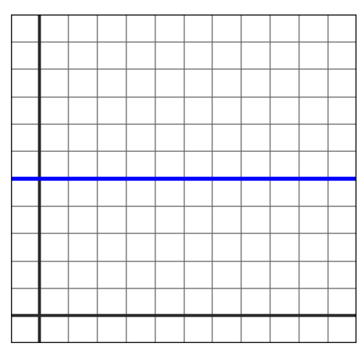
Definition 5.4: "Let *f* be continuous on [*a*, *b*]. The **definite integral of *f* from *a* to *b*** is the unique number *I* which the Riemann sums approach…This number is denoted by $\int_a^b f(x)\,dx$."

$\int$ is the **integral sign**; *a* and *b* are the **limits of integration**; *f*(*x*) is the **integrand**.

Putting Theorem 5.3 and Definition 5.4 together we get $\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(t_k)\,\Delta x_k$.

As was pointed out in Lecture 5.1, if we want to equate a Riemann sum with geometric area, we'd need *f* to be a continuous non-negative function on [*a*, *b*]. In a similar fashion, for a function *f* which is continuous and non-negative on [*a*, *b*], the area of the region *R* between the graph of *f* and the *x*-axis is defined to be $\int_a^b f(x)\,dx$.

Example A: Show that $\int_2^{10} 5\,dx = 40$.



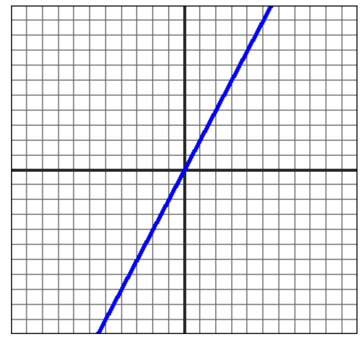
In Example 1, the text uses this same process to show that for any constant function $f(x) = c$, it will always be true that $\int_a^b c\,dx = c(b-a)$, which makes sense because the shape will always be rectangular, with area = height times width.

In Example 2, the text uses the same process (with more complicated algebra) to show that for the linear function $f(x) = x$, it will always be true that $\int_a^b x\,dx = \frac{1}{2}\left(b^2 - a^2\right)$. (In the process, lots of terms zero each other out.)

In the next paragraph, the text points out that it is relatively easy to use a similar process to show that

$$\int_a^b cx\,dx = c\int_a^b x\,dx = \frac{c}{2}\left(b^2 - a^2\right).$$

Example B: Evaluate $\int_2^5 2x\,dx$ and $\int_{-5}^2 2x\,dx$.

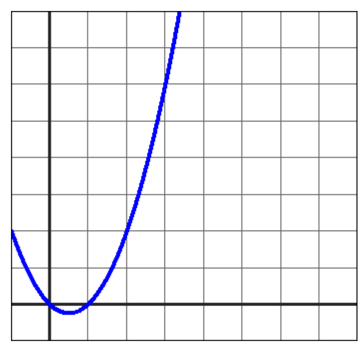


In the first instance, since all *y*-coordinates are positive, our result is "area under the curve".
In the second instance, our result is not a geometric area, since, in geometry, areas are never negative.

Two more tidbits from the text that you may find useful for homework: $\int_a^b cx^2\,dx = c\int_a^b x^2\,dx = \frac{c}{3}\left(b^3 - a^3\right)$ (following Example 2) and $\int_a^b cx^3\,dx = c\int_a^b x^3\,dx = \frac{c}{4}\left(b^4 - a^4\right)$ (text Exercise 36).

Now back to our underlying concept, $\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(t_k)\,\Delta x_k$.

Example C. Approximate $\int_1^3 \left(x^2 - x\right)dx$ by computing $L_f(P)$ and $U_f(P)$ on $P = \{1, 2, 3\}$.



Example D. Approximate $\int_0^2 |x-1|\,dx$ by computing Riemann sums (left, midpoint and right) on a partition having 4 subintervals of the same length.