

## Calculus 140, section 5.3 Special Properties of the Definite Integral

### 7 tidbits about integrals you didn't know you needed to know

notes by Tim Pilachowski

Definition 5.6, part 1: “Let  $f$  be continuous on  $[a, b]$ . Then  $\int_a^a f(x) dx = 0$ .”

It's fairly obvious why this must be true.

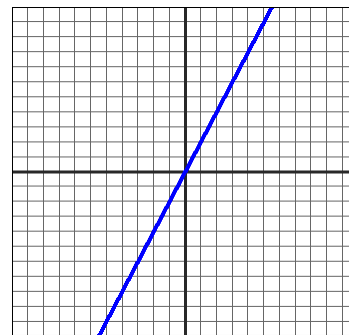
$$\int_a^a f(x) dx = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^1 f(t_k) \Delta x_k = \lim_{\|P\| \rightarrow 0} f(t_k) * 0 = 0$$

In geometric area terms, a line segment, which is infinitely thin, has a width, and therefore an area, equal to 0. Among other things this means that the areas under a curve on the intervals  $[a, b]$ ,  $[a, b)$ ,  $(a, b]$  and  $(a, b)$  are all mathematically equal.

Definition 5.6, part 2: “Let  $f$  be continuous on  $[a, b]$ . Then  $\int_b^a f(x) dx = -\int_a^b f(x) dx$ .”

We've already encountered a similar concept when we looked at distance and velocity: positive is up or forward, and negative is down or backward. The “integral from  $b$  to  $a$ ” is the “integral from  $a$  to  $b$ ” in reverse gear.

5.2 Example B revisited: Evaluate  $\int_2^2 2x dx$  and  $\int_5^2 2x dx$ .



Theorem 5.7, Rectangle Property: “For any numbers  $a, b$ , and  $c$ ,  $\int_a^b c dx = c(b - a)$ .”

Proof:

5.2 Example A revisited: Evaluate  $\int_{10}^2 5 dx$ .

Theorem 5.8, Addition Property: “Let  $f$  be continuous on an interval containing  $a$ ,  $b$ , and  $c$ . Then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.”$$

The text has put the highly-technical proof in the Appendix.

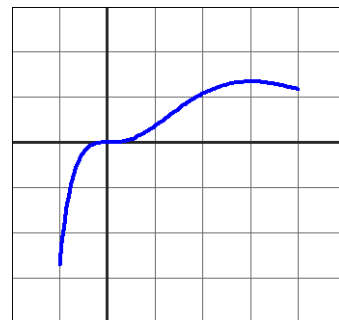
Example C: Given the piecewise continuous function  $f(x) = \begin{cases} 2x & x < 3 \\ 5 & x \geq 3 \end{cases}$ , evaluate  $\int_0^6 f(x) dx$ .

“Piecewise continuous” means the function is composed of pieces, each of which is continuous.  
(See the text for the detailed explanation.)

Theorem 5.9, Comparison Property: “Let  $f$  be continuous on  $[a, b]$ , and suppose  $m \leq f(x) \leq M$  for all  $x$  in  $[a, b]$ . Then  $m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$ .”

The value  $m(b-a)$  is a **lower bound** for the integral;  $M(b-a)$  is an **upper bound** for the integral.  
The text’s proof, based on a Riemann sum, takes 5 lines of text and 3 lines of equations.

4.1 Example C revisited: Using the Comparison Property, find lower and upper bounds for  $\int_{-1}^4 \frac{x^3}{e^x} dx$ .



Corollary 5.10: “Let  $f$  be nonnegative and continuous on  $[a, b]$ . Then  $\int_a^b f(x) dx \geq 0$ .”

Proof:

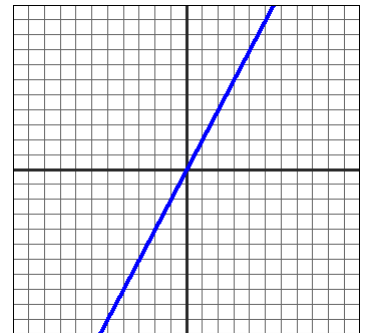
Now we come to the 7<sup>th</sup> and final tidbit of this section.

Theorem 5.11, Mean Value Theorem for Integrals: “Let  $f$  be continuous on  $[a, b]$ . Then there is a number  $c$  in  $[a, b]$  such that  $\int_a^b f(x) dx = f(c)(b - a)$ .”

Proof:

The value  $\frac{\int_a^b f(x) dx}{b - a} = \frac{1}{b - a} \int_a^b f(x) dx$  is called the **mean value** or **average value** of  $f$  on  $[a, b]$ .

5.2 Example B once again: Find the mean value of  $f(x) = 2x$  on  $[2, 5]$ .



interpretation in calculus terms:

interpretation in geometric terms: