Calculus 140, section 5.6 Integration by Substitution

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Now we begin to address integrals which are not as easy as "finding an antiderivative". The first method is called *integration by substitution*, and is like a "chain rule for derivatives" in reverse. (This is the only additional method of integration we'll cover in Math 140. There are others addressed in Math 141.)

Recall that, by the chain rule, $\frac{d}{dx}G[f(x)] = G'[f(x)] * f'(x)$. For an integral that we can recognize as $\int g[f(x)] * f'(x)\,dx = \int G'[f(x)] * f'(x)\,dx$, we can integrate our way back to $G[f(x)] + C$ [Theorem 5.20]. The necessary conditions are that $g \circ f$ and $f'(x)$ must be continuous on an interval *I*, and *G* is an antiderivative of *g*.

The hard part is the recognition of this form.

Example A: Find $\int \frac{2\ln x}{x}\,dx$. *answer*: $(\ln x)^2 + C$

Based on the knowledge of derivatives and antiderivatives covered so far, besides the substitution above there are some general forms of integrals to look for:

$$\int u^n\,du \qquad \int e^u\,du \qquad \int \frac{1}{u}\,du$$

Example B: Evaluate $\int \frac{3}{\sqrt{3x+1}}\,dx$. *answer*: $2(3x+1)^{1/2} + C$

Example B extended: Evaluate $\int \frac{9x^2+2}{\sqrt{3x^3+2x+1}}\,dx$. *answer*: $2\left(3x^3+2x+1\right)^{1/2} + C$

Example C: $\int \frac{x}{e^{x^2}}\,dx$. *answer*: $-\frac{1}{2}e^{-x^2}+C$

Example D: $\int \frac{x^3}{x^4+8}\,dx$. *answer*: $\frac{1}{4}\ln\left(x^4+8\right)+C$.

(Note that domain of ln *x* is not an issue here, since $x^4+8>0$ for all *x*.)

Example E: $\int x^2\cos\left(x^3\right)dx$. *answer*: $\frac{1}{3}\sin\left(x^3\right)+C$

Example F: $\int \sin^2(4x)\cos(4x)\,dx$. *answer*: $\frac{1}{12}\sin^3(4x)+C$

Side note: The text mentions trigonometric identities which allow us to integrate $y=\cos^2 x$ and $y=\sin^2 x$. For your edification, the derivation of these two identities is provided at the end of this lecture outline.

Example G: $\int \frac{x}{\sqrt{2x+3}}\,dx$. *answer*: $\frac{(2x+3)^{3/2}}{6} - \frac{3(2x+3)^{1/2}}{2} + C$

Example H: Find $\int_1^{e^2} \frac{\ln x}{2x}\,dx$. *answer*: 1

Example H: Find $\int_1^{e^2} \frac{\ln x}{2x}\,dx$ using the Change of Variables (Change of Limits) Limits Rule. *answer*: 1

Example I: $\int_{-1}^{2} 5x\,e^{x^2-1}\,dx$. *answer*: $\frac{5}{2}\left(e^3 - 1\right)$

Example J: $\int_0^2 \frac{x}{\left(x^2+3\right)^5}\,dx$. *answer*: $-\frac{1}{8}\left(\frac{1}{7^4}-\frac{1}{3^4}\right) \approx 0.0014911482$

Now, as promised earlier, a derivation of the trigonometric identities mentioned in the text. (These become very important in Math 141.) They both derive from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$.

$$\begin{aligned} \cos^2 x - \sin^2 x &= \cos(2x) \\ \cos^2 x - \left(1-\cos^2 x\right) &= \cos(2x) \\ 2\cos^2 x - 1 &= \cos(2x) \\ 2\cos^2 x &= 1+\cos(2x) \\ \cos^2 x &= \frac{1}{2}+\frac{1}{2}\cos(2x) \end{aligned}$$

$$\begin{aligned} \cos^2 x - \sin^2 x &= \cos(2x) \\ \left(1-\sin^2 x\right) - \sin^2 x &= \cos(2x) \\ 1 - 2\sin^2 x &= \cos(2x) \\ -2\sin^2 x &= -1+\cos(2x) \\ \sin^2 x &= \frac{1}{2}-\frac{1}{2}\cos(2x) \end{aligned}$$

Thus, $\int \cos^2 x\,dx = \int \frac{1}{2}+\frac{1}{2}\cos(2x)\,dx$ and $\int \sin^2 x\,dx = \int \frac{1}{2}-\frac{1}{2}\cos(2x)\,dx$, each of which involves the simple substitution $u = 2x$.