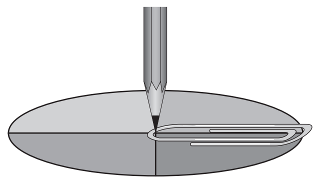
16

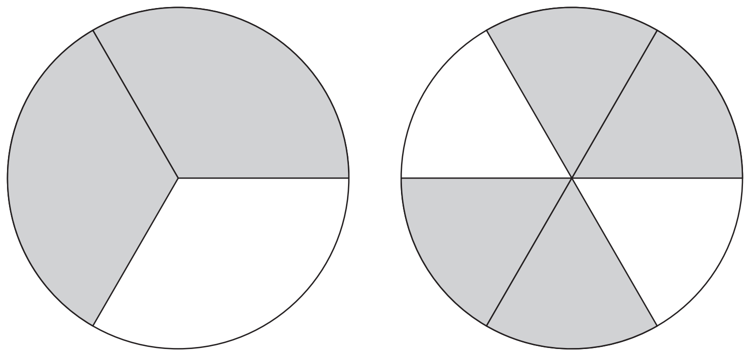
Probability

16.1 Basic Principles of Probability

Class Activity 16A: Probabilities with Spinners



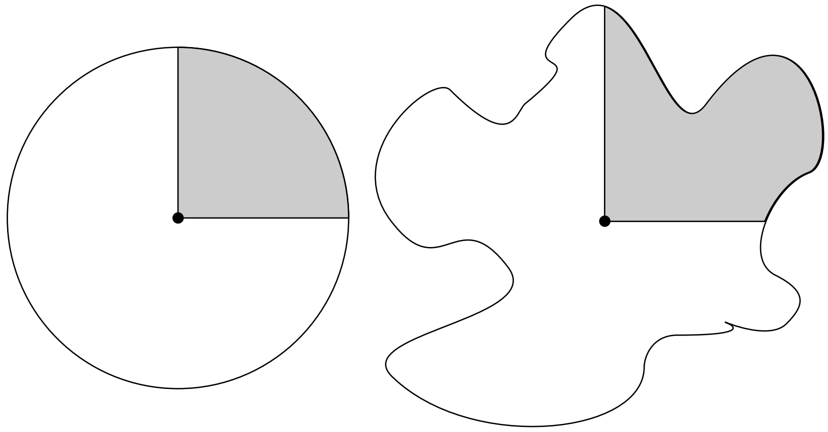
1. Many children's games use "spinners." You can make a simple spinner by placing the tip of a pencil through a paper clip and holding the pencil so that its tip is at the center of the circle as shown. The paper clip should spin freely around the pencil tip.



This problem is similar to a problem in an activity book for grades 1–3; see [6]:

Compare the two spinners shown above. For which spinner is a paper clip most likely to point into a shaded region? Explain your answer.

2. Compare the next two spinners. For which spinner is the paper clip that spins around a pencil point held at the indicated center point most likely to land in a shaded region? Explain your answer.



3. Draw a 4-color spinner (red, green, yellow, blue) such that

- landing on green is twice as likely as landing on red;
- landing on yellow is equally likely as landing on green;
- landing on blue is more likely than landing on yellow.

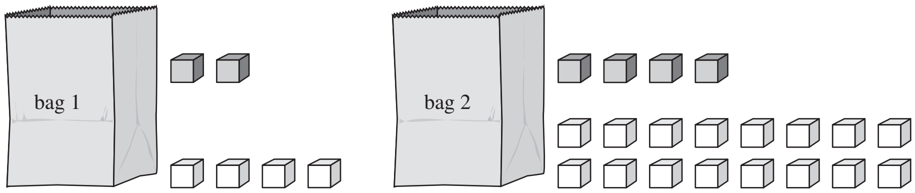
Determine the probabilities of landing on each of the colors on your spinner and explain your reasoning.

Could someone else make a different spinner with different probabilities?

Class Activity 16B: Some Probability Misconceptions

1. Kevin has a bag that is filled with 2 red balls and 1 white ball. Kevin says that because there are two different colors he could pick from the bag, the probability of picking the red ball is $\frac{1}{2}$. Is this correct?

2. A family math night at school features the following game. There are two opaque bags, each containing red blocks and yellow blocks. Bag 1 contains 2 red blocks and 4 yellow blocks. Bag 2 contains 4 red blocks and 16 yellow blocks. To play the game, you pick a bag and then you pick a block out of the bag without looking. You win a prize if you pick a red block. Eva thinks she should pick from bag 2 because it has more red blocks than bag 1. Is Eva more likely to pick a red block if she picks from bag 2 than from bag 1? Why or why not?



3. The probability of winning a game is $\frac{3}{1000}$. Does this mean that if you play the game 1000 times, you will win 3 times? If not, what does the probability of $\frac{3}{1000}$ stand for?

Class Activity 16C: Using Experimental Probability to Make Predictions

A family math night at school includes the following activity. A bag is filled with 10 small counting bears that are identical except that 4 are yellow and the rest are blue. A sign next to the bag gives instructions for the activity:

> Win a prize if you guess the correct number of yellow bears in the bag! There are 10 bears in the bag. Some are yellow and the rest are blue. Here's what you do:
>
> - Reach into the bag, mix well, and pick out a bear.
> - Get a sticky note that is the same color as your bear, write your name and your guess on the note and add your note to the others of the same color.
> - Put your bear back in the bag, and mix well.

The sticky notes will be organized into columns of 10, so it will be easy to count up how many of each there are.

1. How will students be able to use the results of this activity to estimate the number of yellow bears in the box?

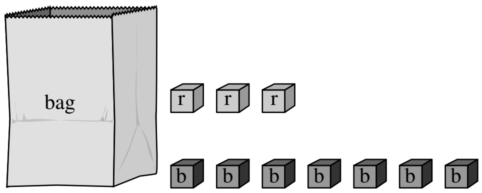
2. What do you expect will happen as the night goes on and more and more bears are picked?

3. Discuss any additions or modifications you would like to make to the activity if you were going to use it for math night at your school.

Class Activity 16D:

Experimental versus Theoretical Probability: Picking Cubes from a Bag

Each person (or small group) will need an opaque bag, 3 red cubes, 7 blue cubes, and a sticky note. In this activity you will compare experimental and theoretical probabilities of picking a red cube from a bag containing 3 red and 7 blue cubes.



1. Put the 10 cubes in the bag, mix them up, and randomly pick a cube from the bag without looking. Record the color of the cube, and put the cube back in the bag. Repeat until you have picked 10 cubes. Record the number or red cubes you picked on your sticky note. Calculate the experimental probability of picking a red cube based on your 10 picks. Is it the same as the theoretical probability of picking red?

2. Now work with a large group (e.g., the whole class). Collect the sticky notes of part 1 from the full group. Determine the total number of reds picked and the total number of picks among the large group. Use these results to determine the experimental probability of picking a red cube obtained by the large group. Compare this experimental probability with the theoretical probability of picking red.

3. Use the sticky notes of part 1 to create a dot plot. How is the fact that there are 3 red cubes and 7 blue cubes in the bag reflected in the dot plot?

Class Activity 16E: If You Flip 10 Pennies, Should Half Come Up Heads?

You will need a bag, 10 pennies or 2-color counters, and some sticky notes for this activity.

1. Make a guess: What do you think the probability is of getting exactly 5 heads on 10 pennies when you dump the 10 pennies out of a bag?

2. Put the 10 pennies in the bag, shake them up, and dump them out. Record the number of heads on a sticky note. Repeat this for a total of 10 times, using a new sticky note each time. Out of these 10 tries, how many times did you get 5 heads? Therefore, what is the experimental probability of getting 5 heads based on your 10 trials?

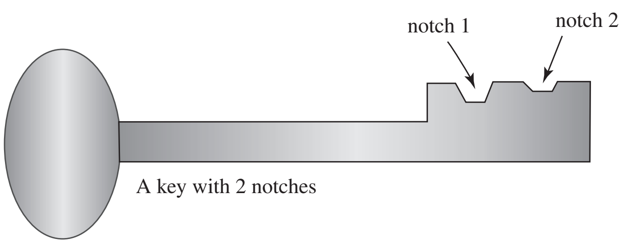
3. Now work with a large group (e.g., the whole class). Collect the whole group's data on the sticky notes from part 2. Find a way to display these data so that you can see how often the whole group got 5 heads and other numbers of heads.

4. Is the probability of getting exactly 5 heads from 10 coins 50%? What does your data display from part 3 suggest?

16.2 Counting the Number of Outcomes

Class Activity 16F: How Many Keys Are There?

Have you ever wondered: how can millions of *different* car keys be produced, even though car keys are not very big? Keys are manufactured to be distinct from one another by the way they are notched. Car keys have intricate notching. For simplicity, in this activity let's consider only simple keys that are notched on one side.



1. Suppose a simple type of key is to be made with 2 notches, and that each notch can be one of 3 depths: deep, medium, or shallow. How many different keys can be made this way? Explain.

2. Explain how to use multiplication to solve the problem in part 1 if you haven't already.

3. Now suppose the key is to be made with 4 notches, and each notch can be one of 3 depths: deep, medium, or shallow. How many keys can be made this way? Explain.

4. Now suppose the key is to be made with 10 notches, and each notch can be one of 5 depths. How many keys can be made this way? Explain.

Class Activity 16G: Counting Outcomes: Independent versus Dependent

1. How many 3-letter security codes can be made from the 4 letters A, B, C, D? For example, BAB and ABB are two such codes, and DAC is another. Explain.

2. How many 3-letter security codes can be made from the 4 letters A, B, C, D without using a letter twice? For example, BAC and ADB are two such codes. Explain.

3. Explain how to use multiplication to solve the problem in part 2 if you haven't already.

4. Contrast how you use multiplication to solve the problems in parts 1 and 2 and explain the distinction.

16.3 Calculating Probabilities in Multi-Stage Experiments

Class Activity 16H: Number Cube Rolling Game

Maya, James, Kaitlyn, and Juan are playing a game in which they take turns rolling a pair of number cubes. Each child has chosen a "special number" between 2 and 12, and each child receives 8 points whenever the total number of dots on the two number cubes is their special number. (They receive their points regardless of who rolled the number cubes. Their teacher picked 8 points so that the children would practice counting by 8s.)

- Maya's special number is 7.
- James's special number is 10.
- Kaitlyn's special number is 12.
- Juan's special number is 4.

The first person to get to 100 points or more wins. The children have played several times, each time using the same special numbers. They notice that Maya wins most of the time. They are wondering why.

1. Roll a pair of number cubes many times, and record the total number of dots each time. Display your data so that you can compare how many times each possible number between 2 and 12 has occurred. What do you notice?

2. Draw an array showing all possible outcomes on each number cube when a pair of number cubes are rolled. (Think of the pair as *number cube 1* and *number cube 2*.)
   a. For which outcomes is the total number of dots 7? 10? 12? 4?
   b. What is the probability of getting 7 total dots on a roll of two number cubes? What is the probability of getting 10 total dots on a roll of two number cubes? What about for 12 and 4?
   c. Is it surprising that Maya kept winning?

Class Activity 16I:
Picking Two Marbles from a Bag of 1 Black and 3 Red Marbles

You will need an opaque bag, 3 red marbles, and 1 black marble for this activity. Put the marbles in the bag.

If you reach in without looking and randomly pick out 2 marbles, what is the probability that 1 of the 2 marbles you pick is black? You will study this question in this activity.

1. Before you continue, make a guess: What do you think the probability of picking the black marble is when you randomly pick 2 marbles out of the 4 marbles (3 red, 1 black) in the bag?

2. Pick 2 marbles out of the bag. Repeat this many times, recording what you pick each time. What fraction of the times did you pick the black marble?

3. Now calculate the probability theoretically, using a tree diagram. For the purpose of computing the probability, think of first picking one marble, then (without putting this marble back in the bag) picking a second marble. From this point of view, draw a tree diagram that will show all possible outcomes for picking the two marbles. But draw this tree diagram in a special way, *so that all outcomes shown by your tree diagram are equally likely.*

Hints: The first stage of the tree should show all possible outcomes for your first pick. Remember that all branches you show should be equally likely. In the second stage, the branches you draw should depend on what happened in the first stage. For instance, if the first pick was the black marble, then the second pick must be one of the three red marbles.

a. How many total outcomes for picking 2 marbles, 1 at a time, out of the bag of 4 (3 red, 1 black) does your tree diagram show?

b. In how many outcomes is the black marble picked (on 1 of the 2 picks)?

c. Use your answers to parts 3 (a) and (b) and the basic principles of probability to calculate the probability of picking the black marble when you pick 2 marbles out of a bag filled with 1 black and 3 red marbles.

4. Why was it important to draw the tree diagram so that all outcomes were equally likely?

5. Here's another method for calculating the probability of picking the black marble when you pick 2 marbles out of a bag filled with 1 black and 3 red marbles:

a. How many unordered pairs of marbles can be made from the 4 marbles in the bag?

b. How many of those pairs of marbles in part (a) contain the black marble? (Use your common sense.)

c. Use parts a and b and basic principles of probability to determine the probability of picking the black marble when you pick 2 marbles out of a bag containing 1 black and 3 red marbles.

6. Compare your answers to parts 3a and 5a, and compare your answers to 3b and 5b. How and why are they different?

Class Activity 16J: More Probability Misconceptions

1. Simone has been flipping a coin and has just flipped 5 heads in a row. Simone says that because she has just gotten so many heads, she is more likely to get tails than heads the next time she flips. Is Simone correct? What is the probability that Simone's next flip will be a tail? Does the answer depend on what the previous flips were?

2. Let's say you flip 2 coins simultaneously. There are 3 possible outcomes: Both are heads, both are tails, or one is heads and the other is tails. Does this mean that the probability of getting one head and one tail is $\frac{1}{3}$?

Class Activity 16K: Expected Earnings from the Fall Festival

Ms. Wilkins is planning a game for her school's fall festival. She will put 2 red, 3 yellow, and 10 green plastic bears in an opaque bag. (The bears are identical except for their color.) To play the game, a contestant will pick 2 bears from the bag, one at a time, without putting the first bear back before picking the second bear. Contestants will not be able to see into the bag, so their choices are random. To win a prize, the contestant must pick a green bear first and then a red bear. The school is expecting about 300 people to play the game. Each person will pay 50 cents to play the game. Winners receive a prize that costs the school $2.

1. How many prizes should Ms. Wilkins expect to give out? Explain.

2. How much money (net) should the school expect to make from Ms. Wilkins's game? Explain.

16.4 Using Fraction Arithmetic to Calculate Probabilities

Class Activity 16L:

Using the Meaning of Fraction Multiplication to Calculate a Probability

Use the circle in Figure 16L.1, a pencil, and a paper clip to make a spinner as follows: Put the pencil through the paper clip, and put the point of the pencil on the center of the circle. The paper clip will now be able to spin freely around the circle.

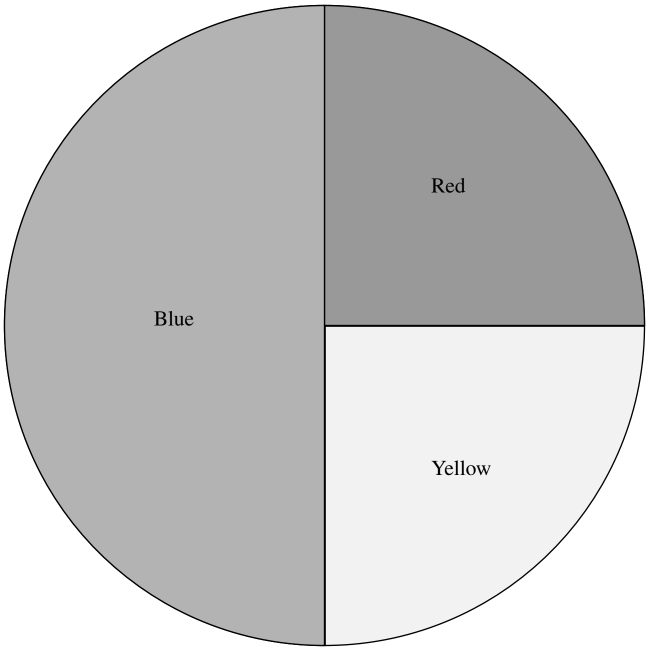
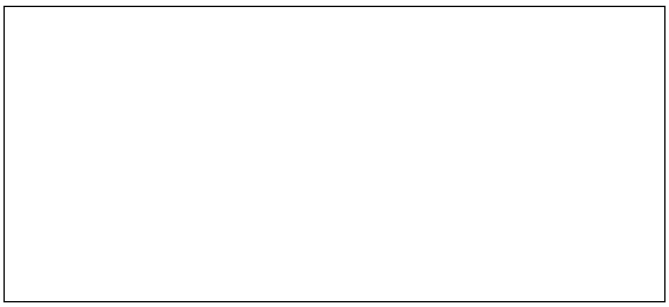


Figure 16L.1 A spinner



This rectangle represents many pairs of spins.

To win a game, Jill needs to spin a blue followed by a red in her next 2 spins.

1. What do you think Jill's probability of winning is? (Make a guess.)

2. Carry out the experiment of spinning the spinner twice in a row 20 times. (In other words, spin the spinner 40 times, but each experiment consists of 2 spins.) Out of those 20 times, how often does Jill win? What fraction of 20 does this represent? Is this close to your guess in part 1?

3. Calculate Jill's probability of winning theoretically as follows: Imagine that Jill carries out the experiment of spinning the spinner twice in a row many times. In the ideal, what fraction of those times should the first spin be blue? _______________ Show this by shading the rectangle on the previous page.

   In the ideal, what fraction of those times when the first spin is blue should the second spin be red? _____________ Show this by further shading the rectangle on the previous page.

   In the ideal, what fraction of pairs of spins should Jill spin first a blue and then a red? Therefore, what is Jill's probability of winning? _________ Explain how you can determine this fraction from the shading of the rectangle and from the meaning of fraction multiplication. Compare your answer with parts 1 and 2.

Class Activity 16M: Using Fraction Multiplication and Addition to Calculate a Probability

A paper clip, an opaque bag, and blue, red, and green tiles would be helpful.

A game consists of spinning the spinner in Figure 16L.1 and then picking a small tile from a bag containing 1 blue tile, 3 red tiles, and 1 green tile. (All tiles are identical except for color, and the person picking a tile cannot see into the bag, so the choice of a tile is random.) To win the game, a contestant must pick the same color tile that the spinner landed on. So a contestant wins from either a blue spin followed by a blue tile or a red spin followed by a red tile.

1. Make a guess: What do you think the probability of winning the game is?

2. If the materials are available, play the game a number of times. Record the number of times you play the game (each game consists of both a spin *and* a pick from the bag), and record the number of times you win. What fraction of the time did you win? How does this compare with your guess in part 1?

3. To calculate the (theoretical) probability of winning the game, imagine playing the game many times. Answer the next questions in order to determine the probability of winning the game.

a. In the ideal, what fraction of the time should the spin be blue? ________ Show this by shading the rectangle below.

In the ideal, what fraction of those times when the spin is blue should the tile that is chosen be blue? ________ Show this by further shading the rectangle below.

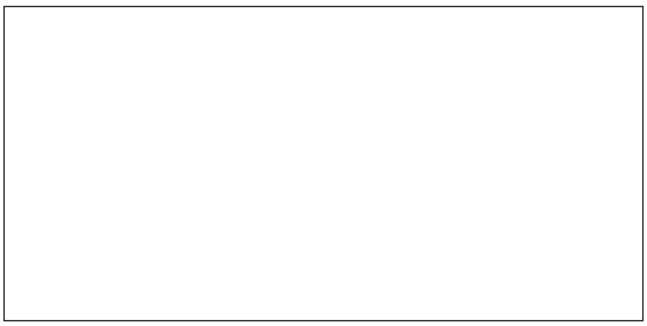
Therefore, in the ideal, what fraction of the time is the spin blue and the tile blue? ________ Explain how you can determine this fraction from the shading of the rectangle and from the meaning of fraction multiplication.

b. In the ideal, what fraction of the time should the spin be red? ________ Show this by shading the rectangle below.

In the ideal, what fraction of those times when the spin is red should the tile that is chosen be red? ________ Show this by further shading the rectangle below.

Therefore, in the ideal, what fraction of the time is the spin red and the tile red? ________ Explain how you can determine this fraction from the shading of the rectangle and from the meaning of fraction multiplication.

c. In the ideal, what fraction of the time should you win the game, and therefore, what is the probability of winning the game? Explain why you can calculate this answer by multiplying and adding fractions. Compare your answer with parts 1 and 2.



This rectangle represents playing the game many times.