Calculus 241, section 11.1 Cartesian Coordinates in Space

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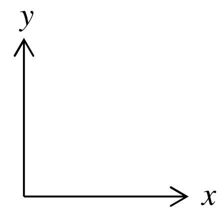
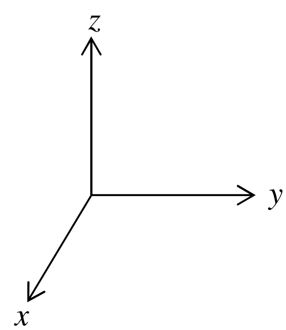
You are already familiar with the two dimensional Cartesian coordinate system, first introduced to you back in Algebra I. Points were plotted using (x, y) coordinates, with x giving us horizontal movement (positive to the right and negative to the left), with y giving us vertical movement (positive moving up and negative moving down).



Note that these are *rectangular* coordinates. That is, all changes in direction are right-angle turns.

We can follow the same procedure in three dimensions, using three axes, and plotting points by making right angle turns.



The technical definition of a point in a three-dimensional grid is "the intersection of three perpendicular planes".

In the standard orientation pictured above, we can see three coordinate planes. The xy plane is the "floor"; the xz plane is the "left wall"; and the yz plane is the "back wall".

The origin (point O) has coordinates (0, 0, 0), and is the intersection of planes with equations $x = 0$, $y = 0$, and $z = 0$.

Just like the two-dimensional Cartesian plane is divided into four Quadrants by the horizontal and vertical axes, a three-dimensional grid is divided into eight Octants. In the standard orientation pictured above, our point of view is in the first Octant, where the x, y, and z coordinates are all positive.

Side note: Any three distinct points will lie in the same plane. We'll spend time later finding the equation of a plane given the (x, y, z) coordinates of three points.

Definition and formula: The distance between two points $P = (x_0, y_0, z_0)$ and $Q = (x_1, y_1, z_1)$ in a three-dimensional space is given by

$$|PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2} .$$

This formula can be verified with two successive applications of the Pythagorean formula. (Rather than duplicate it here in Lecture, I'll refer you to text Figure 11.5.)

Let's explore this notion of *distance*. First of all, recall the basic definition of absolute value of a real number.

For $r \in \Re$, $| r |$ = distance from 0 to r, where distance ≥ 0.

This is, of course, distance in a one-dimensional world.

In two dimensions, the Pythagorean formula gives us the distance d between two points $P = (x_0,\, y_0)$ and $Q = (x_1,\, y_1)$ as

$$d = |PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}\ .$$

So, our definition and formula for distance in three dimensions makes intuitive sense, and fits with Mathematics previously encountered.

Note also that, in three dimensional space, if $Q = O$, then

$$|PO| = |OP| = \sqrt{(x_0 - 0)^2 + (y_0 - 0)^2 + (z_0 - 0)^2} = \sqrt{(x_0)^2 + (y_0)^2 + (z_0)^2}\ .$$

(This observation will come in handy later on.)

Example A. Given $P = (3, 2, 1)$ and $Q = (-1, 4, -5)$, find $|OP|, |PQ|, |QO|$.

Example A extended. What is the perimeter of the triangle which has points O, P and Q at its vertices?

We have three basic properties (laws) governing distance in three dimensional space. Given two points P and Q,

$$|PQ| = 0 \quad \Leftrightarrow \quad P = Q$$

$$|PQ| = |QP|$$

for any third point R, $|PQ| \le |PR| + |RQ|$.

The third law [the triangle inequality] is proved in section 11.3, exercise #25.

In the two-dimensional *xy* plane, what is the definition of a circle?

What is the equation of a circle with center (x_0, y_0) and radius a?

Note that this equation would give us points located on the circle only, and not the area inside the circle. If we want to include the interior area along with the circle itself [closed disk], what would the equation be?

If we want to include the interior area, but not the circle [open disk], what would the equation be?

Now, let's extrapolate to three dimensions. What is the name of the shape in three-dimensional space that is analogous to a circle from a two-dimensional plane?

What is the definition of a sphere?

What is the equation of a sphere with center (x_0, y_0, z_0) and radius a?

Note that this equation would give us points located on the sphere only, and not the space inside the sphere. If we wanted want to include the interior space along with the circle itself [closed ball], what would the equation be?

If we wanted to include the interior space, but not the sphere [open ball], what would the equation be?

Example B. Show that $x^2 + y^2 + z^2 = 6x - 4y + 2z + 2$ is a sphere and find its equation and the coordinates of its center.

Example B extended: What would the equation be if we want the closed ball? The open ball?