Calculus 241, section 11.2 Vectors in Space

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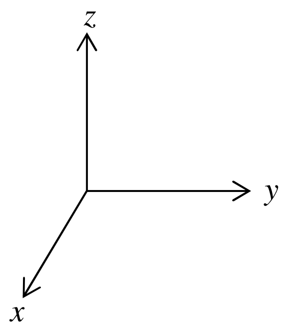
Back to three-dimensional (3-D) space, and our 3-D grid (shown in standard orientation below).



Consider two points, $P = (3, 2, 1)$ and $Q = (-1, 4, -5)$.

If we connect these points with an arrow pointing from P to Q, we have a visual representation of a **vector**.

More formally: "A **vector** is an ordered triple (a_1, a_2, a_3) of numbers. The numbers $a_1, a_2,$ and a_3 are called the **components** of the vector. The vector $\overrightarrow{PQ}$ associated with the directed line segment with initial point $P = (x_0, y_0, z_0)$ and terminal point $Q = (x_1, y_1, z_1)$ is $(x_1 - x_0, y_1 - y_0, z_1 - z_0)$." [Definition 11.1]

Example A. Consider points $P = (3, 2, 1)$ and $Q = (-1, 4, -5)$. Identify the vectors $\overrightarrow{PQ}$ and $\overrightarrow{OP}$.

Note that even though the expressions of $\overrightarrow{OP} = (3, 2, 1)$ and $P = (3, 2, 1)$ look the same, $\overrightarrow{OP}$ expresses the components of a vector and P gives us the coordinates of a point. You *must* pay attention to **context**!

Two vectors (a_1, a_2, a_3) and (b_1, b_2, b_3) are equal if and only if (iff) $a_1 = b_1, a_2 = b_2,$ and $a_3 = b_3$.

Example A extended. Given $R = (-3, 5, -2)$ and $S = (-7, 9, -8)$, and P and Q as given above, determine whether $\overrightarrow{RS} = \overrightarrow{PQ}$.

Uses of vectors include

a velocity vector has both direction and magnitude (= speed)

a force vector has both direction and magnitude

(Remember these from Calc I and Calc II respectively?)

A quick observation about the notation for vectors.

In print, vectors will often be written as bold-print letters, e.g. **a**, **b**, **c**, to distinguish them from the components of a vector *a*, *b*, and *c*. In writing, vectors should be written with arrows: $\vec{a}, \vec{b}, \vec{c}$.

The zero vector $(0, 0, 0) = \mathbf{0} = \vec{0}$.

Side note: Any two distinct, connected vectors will lie in the same plane. (Connected = common starting point and/or ending point.) We'll spend time later finding the equation of a plane given two connected vectors.

Definition 11.2: The **length (or norm)** of a vector **a** = (a_1, a_2, a_3) is ||**a**|| = $\|\vec{a}\| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$.

Example A again. $P = (3, 2, 1)$ and $Q = (-1, 4, -5)$. Find $\|\overrightarrow{PQ}\|$.

Note that, in general, given any two points P and Q, the length (norm) of the vector, $\|\overrightarrow{PQ}\|$, = distance between the two points, $|PQ|$.

Definition: A **unit vector** has length (norm) = 1.
We will denote three special unit vectors, corresponding to the three axes of our 3-D grid:

$$\vec{i} = (1, 0, 0),\ \vec{j} = (0, 1, 0),\ \vec{k} = (0, 0, 1).$$

Is it obvious that the lengths of **i**, **j**, and **k** all equal 1?

Definition 11.3: We now define three arithmetic combinations of vectors: sum, difference, and scalar multiple = $c\mathbf{a} = c\,\vec{a}$.

Example B. Given $\vec{a} = (4, 2, -6),\ \vec{b} = (1, 3, 5)$, find **a** + **b**, **a** – **b**, 2**a**.

We can give a geometric interpretation to these three arithmetic combinations of vectors.

Vector arithmetic has "laws" (identities) similar to those of elementary school arithmetic:
additive & multiplicative identities, commutativity, associativity, distribution of multiplication over addition.

$$\mathbf{0} + \mathbf{a} = \vec{0} + \vec{a} = \mathbf{a} + \mathbf{0} = \vec{a} + \vec{0} = \mathbf{a} \qquad \mathbf{a} + \mathbf{b} = \vec{a} + \vec{b} = \mathbf{b} + \mathbf{a} = \vec{b} + \vec{a}$$

$$\mathbf{a} - \mathbf{b} = \vec{a} - \vec{b} = \mathbf{a} + (-1)\mathbf{b} = \vec{a} + (-1)\vec{b} \qquad (\mathbf{a} + \mathbf{b}) + \mathbf{c} = (\vec{a} + \vec{b}) + \vec{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) = \vec{a} + (\vec{b} + \vec{c})$$

$$0\mathbf{a} = 0\vec{a} = \mathbf{0} = \vec{0} \qquad 1\mathbf{a} = 1\vec{a} = \mathbf{a} = \vec{a}$$

$$c(\mathbf{a} + \mathbf{b}) = c\,(\vec{a} + \vec{b}) = c\mathbf{a} + c\mathbf{b} = c\,\vec{a} + c\,\vec{b}$$

Using the arithmetic operations above, any vector $\mathbf{a} = \vec{a}$ can be expressed as a combination of unit vectors $\mathbf{i} = \vec{i}$, $\mathbf{j} = \vec{j}$, $\mathbf{k} = \vec{k}$, which will be very useful as we progress through the semester.

Example B revisited. Given $\vec{a} = (4, 2, -6),\ \vec{b} = (1, 3, 5)$, express **a**, **a** + **b**, and 2**a** as arithmetic combinations of unit vectors **i**, **j**, and **k**.

In general, we can say that

$$\vec{a}+\vec{b}=(a_1+b_1)\vec{i}+(a_2+b_2)\vec{j}+(a_3+b_3)\vec{k} \quad \text{and}$$
$$c(\vec{a})=c\left(a_1\vec{i}+a_2\vec{j}+a_3\vec{k}\right)=c\,a_1\vec{i}+c\,a_2\vec{j}+c\,a_3\vec{k}\,.$$

Note also that $\|\vec{a}\|=\left\|a_1\vec{i}+a_2\vec{j}+a_3\vec{k}\right\|=\sqrt{(a_1)^2+(a_2)^2+(a_3)^2}$. [definition/formula for norm of a vector]

Also, we have $\|c\,\vec{a}\|=\left\|c\,a_1\vec{i}+c\,a_2\vec{j}+c\,a_3\vec{k}\right\|=\sqrt{(c\,a_1)^2+(c\,a_2)^2+(c\,a_3)^2}=|c|\,\|\vec{a}\|$.

By implication, when multiplied by a scalar $c > 0$, the direction of a vector does not change, but the norm (length/magnitude) is multiplied by c. If we think in terms of a velocity vector, the object in motion is still moving along the same path, but with an increase or decrease in speed.

When multiplied by a scalar $c < 0$, the direction is reversed, and the magnitude is multiplied by $|c|$.

We can give a geometric interpretation to scalar multiplication and addition of vectors, $c\mathbf{a} + \mathbf{b}$.

Definition 11.4: Two nonzero vectors $\vec{a}$ and $\vec{b}$ are **parallel** iff there is a scalar c such that $\vec{b}=c\,\vec{a}$.

Example C. Given $\vec{a}=4i-6j+2k,\ \vec{b}=-6i+9j-3k$, determine whether or not **a** and **b** are parallel.

Definition: The **unit vector in the direction of a non-zero vector** $\vec{a}$ is $\dfrac{\vec{a}}{\|\vec{a}\|}$.

Example C extended. Given $\vec{a}=4i-6j+2k$, find the unit vector in the direction of **a**.

Recall the 2-D Cartesian grid encountered in Calc I and Calc II, and rectangular vs. polar coordinates. We can think of a vector in xy plane as a directed line segment connecting the origin to a point, and use trigonometric functions and/or polar coordinates to express the vector. We'd have θ (in radians) as the angle between the horizontal axis and the vector, r as the norm of the vector, and $r\cos\theta$ and $r\sin\theta$, respectively, as the **i** and **j** components of the vector. (See text example 7 and practice execise # 28.)

Recall from earlier: Uses of vectors include force vectors, which have direction, and magnitude.

Example D. Stan & Oliver are moving a piano (on wheels) by tugging on ropes attached to it. Stan is holding his 5 foot rope at a height of 3 feet above ground and 3 feet to the left of the piano, exerting a force of 6 pounds. Oliver is holding his 4 foot rope at a height of 2 feet above ground and 2 feet to the right of the piano, exerting a force of 4 pounds. Find the resulting force being exerted on the piano.