Calculus 241, section 11.3 Dot Product

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Here's a basic question: Now that we have sum, difference and scalar multiplication of vectors, is there a way to describe vector times vector multiplication? The answer is, "Yes." In fact, there are two ways.

Definition 11.5: Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ be two vectors.

The **dot product** (or **scalar product** or **inner product**) of $\vec{a}$ and $\vec{b}$ = $\vec{a} \bullet \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Example A. Given $\vec{a} = 4\vec{i} + 2\vec{j} - 6\vec{k}$ and $\vec{b} = \vec{i} + 3\vec{j} - 5\vec{k}$, find $\vec{a} \bullet \vec{b}$.

Important note: The dot product of two vectors is not a vector but a number (scalar).

The expected properties apply to dot products.

Identities: $\vec{a} \bullet \vec{i} = \vec{i} \bullet \vec{a} = a_1$ $\quad \vec{a} \bullet \vec{j} = \vec{j} \bullet \vec{a} = a_2$ $\quad \vec{a} \bullet \vec{k} = \vec{k} \bullet \vec{a} = a_3$

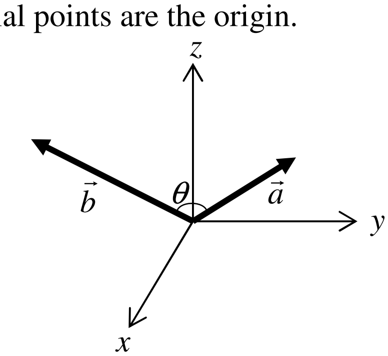
$\vec{i} \bullet \vec{j} = \vec{j} \bullet \vec{k} = \vec{k} \bullet \vec{i} = 0$ $\quad \vec{i} \bullet \vec{i} = \vec{j} \bullet \vec{j} = \vec{k} \bullet \vec{k} = 1$

Commutativity: $\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$

Associativity: $(c\vec{a}) \bullet \vec{b} = c(\vec{a} \bullet \vec{b}) = \vec{a} \bullet (c\vec{b})$

Distribution: $\vec{a} \bullet (\vec{b} + \vec{c}) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c}$, $(\vec{a} + \vec{b}) \bullet \vec{c} = \vec{a} \bullet \vec{c} + \vec{b} \bullet \vec{c}$

Definition: The **angle** between two nonzero vectors $\vec{a}$ and $\vec{b}$ is defined to be the angle θ, $0 \le \theta \le \pi$, formed by the directed line segments whose initial points are the origin.



Theorem 11.6. Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ be two non-zero vectors, and let θ be the angle between $\vec{a}$ and $\vec{b}$. Then, $\vec{a} \bullet \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$.

The text does the proof using the Law of Cosines.

To find the angle between nonzero vectors, use $\vec{a} \bullet \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$ and solve for θ.

Example B. (11.1 A revisited). Given $P = (3, 2, 1)$ and $Q = (-1, 4, -5)$, verify $\overrightarrow{PQ} \bullet \overrightarrow{OQ} = \left\| \overrightarrow{PQ} \right\| \left\| \overrightarrow{OQ} \right\| \cos \frac{\pi}{6}$ and $\overrightarrow{OQ} \bullet \overrightarrow{OP} = \left\| \overrightarrow{OQ} \right\| \left\| \overrightarrow{OP} \right\| \cos \frac{\pi}{2}$.

The second dot product above illustrates Corollary 11.7, part a.

Corollary 11.7: a. Two nonzero vectors **a** and **b** are perpendicular iff $\vec{a} \bullet \vec{b} = 0$.

b. For any vector $\vec{a}$, $\vec{a} \bullet \vec{a} = \| \vec{a} \|^2$, or equivalently $\| \vec{a} \| = \sqrt{\vec{a} \bullet \vec{a}}$

To prove that two nonzero vectors are perpendicular, it is sufficient to show that $\vec{a} \bullet \vec{b} = 0$.

Definition 11.8. Let $\vec{a}$ be a nonzero vector. The **projection** of a vector $\vec{b}$ onto $\vec{a}$ is the vector $\overrightarrow{\text{pr}_a b}$ defined by $\overrightarrow{\text{pr}_a b} = \left(\frac{\vec{a} \bullet \vec{b}}{\| \vec{a} \|^2} \right) \vec{a}$.

Note that the projection of $\vec{b}$ onto $\vec{a}$ is a scalar multiple of $\vec{a}$.

Example A extended. Let $\vec{a} = 4\vec{i} + 2\vec{j} - 6\vec{k}$ and $\vec{b} = 1\vec{i} + 3\vec{j} - 5\vec{k}$. Find $\overrightarrow{\text{pr}_a b}$.

Side note: The text develops a simple formula for the norm of a projection, $\left\| \overrightarrow{\text{pr}_a b} \right\| = \left(\frac{\left| \vec{a} \bullet \vec{b} \right|}{\| \vec{a} \|^2} \right) \| \vec{a} \| = \frac{\left| \vec{a} \bullet \vec{b} \right|}{\| \vec{a} \|}$.

If two vectors $\vec{a}$ and $\vec{a}'$ are perpendicular, then any nonzero vector $\vec{b}$ lying in the same plane as $\vec{a}$ and $\vec{a}'$ can be **resolved** into vectors parallel to $\vec{a}$ and $\vec{a}'$ with the formula

$$\vec{b} = \overrightarrow{\text{pr}_a b} + \overrightarrow{\text{pr}_{a'} b}$$

Example C. Given $\vec{a} = 4\vec{i} + 2\vec{j}$, $\vec{a}' = \vec{i} - 2\vec{j}$, and $\vec{b} = \vec{i} + 3\vec{j}$, resolve $\vec{b}$ into vectors parallel to $\vec{a}$ and $\vec{a}'$.

See the text for an explanation and Example 7 involving work done by a constant force **F** on an object moving along a line from *P* to *Q*. The solution depends upon resolving **F** into a vector parallel to $\overrightarrow{PQ}$.