Calculus 241, section 11.4 Cross Product

notes by Tim Pilachowski

We have vector addition, subtraction, scalar multiplication and dot product.
Now we come to another way to "multiply vector times vector": cross product.

Definition 11.9: Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ be two vectors. The **cross product** of $\vec{a}$ and $\vec{b}$ is defined by

$$= \vec{a}\times\vec{b} = (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}\,.$$

The text notes that the right-hand side can be considered as the determinant of a 3×3 matrix.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The text then advises that this determinant can be evaluated in the following manner,

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\vec{k}$$

where each 2×2 square matrix determinant is evaluated as $\begin{vmatrix} p & q \\ r & s \end{vmatrix} = ps - qr$, resulting in the formula above.

[See Example 1 and Example 2 in the text for this approach put into practice.]
I offer another method for your consideration. Use whatever works best for you.

1) Start with the nine entries in the determinant, then duplicate the first two columns to the right.
2) Find the product of the 3 entries of all six diagonals, multiplying the positive-slope diagonals times (– 1).
3) Add the six results (i.e. combine like terms).

Example A. Given $\vec{a} = 4\vec{i} + 2\vec{j} - 6\vec{k}$ and $\vec{b} = 1\vec{i} + 3\vec{j} - 5\vec{k}$, find $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$.
answer: $\vec{a}\times\vec{b} = 8\vec{i} + 14\vec{j} + 10\vec{k}$

Note that $\vec{a}\times\vec{b} = -\left(\vec{b}\times\vec{a}\right)$, which follows from the definition of cross product.
Important Note: While the dot product of two vectors is number (scalar), the cross product is another vector.

Properties of the cross product of vectors:

$$\vec{i}\times\vec{j}=\vec{k},\ \vec{j}\times\vec{k}=\vec{i},\ \vec{k}\times\vec{i}=\vec{j} \qquad \text{[text Example 1]}$$

$$\vec{a}\times\vec{a}=\vec{0} \qquad \text{[note: vector } \mathbf{0}\text{]}$$

$$(c\,\vec{a})\times\vec{b}=c\left(\vec{a}\times\vec{b}\right)=\vec{a}\times\left(c\,\vec{b}\right) \qquad \text{[Associativity]}$$

$$\vec{a}\times\left(\vec{b}+\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right) \qquad \text{[Distribution]}$$

$$\left(\vec{a}+\vec{b}\right)\times\vec{c}=\left(\vec{a}\times\vec{c}\right)+\left(\vec{b}\times\vec{c}\right) \qquad \text{[Distribution]}$$

Note that for **triple vector products**, it is usually true that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)\neq\left(\vec{a}\times\vec{b}\right)\times\vec{c}$. (Rather than memorize a formula for a triple vector product, I recommend doing first one cross product, then the other.)

Theorem 11.10 Let $\vec{a}=a_1\,\vec{i}+a_2\,\vec{j}+a_3\,\vec{k}$ and $\vec{b}=b_1\,\vec{i}+b_2\,\vec{j}+b_3\,\vec{k}$ be two non-zero vectors.

a. Then $\vec{a}\bullet\left(\vec{a}\times\vec{b}\right)=0$ and $\vec{b}\bullet\left(\vec{a}\times\vec{b}\right)=0$. [Note this is the ***number*** 0.]

Consequently, if $\vec{a}\times\vec{b}\neq\vec{0}$, then $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. [See section 11.3 Corollary 11.7.]

Side observation: If $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, then it will also be perpendicular to the plane determined by $\vec{a}$ and $\vec{b}$. In theory, there are two possible directions for $\vec{a}\times\vec{b}$. The direction can be determined by the **right-hand rule**: Curl the fingers of the right hand from $\vec{a}$ to $\vec{b}$ through θ, and the thumb will point in the direction of $\vec{a}\times\vec{b}$.

b. If θ is the angle between $\vec{a}$ and $\vec{b}$, $0\leq\theta\leq\pi$, then $\left\|\vec{a}\times\vec{b}\right\|=\left\|\vec{a}\right\|\left\|\vec{b}\right\|\sin\theta$.

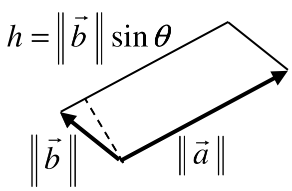
[Contrast section 11.3 Theorem 11.6, $\vec{a}\bullet\vec{b}=\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta$.]

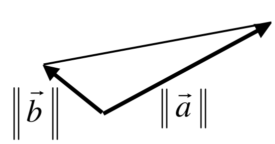
The text does the proof using the definitions of dot product and cross product.

Extending Theorem 11.10b to the case in which $\theta=0$ or $\theta=\pi$, we get Corollary 11.11.
Corollary 11.11: Two nonzero vectors are parallel iff $\vec{a}\times\vec{b}=\vec{0}$. [Note this is the ***vector*** $\vec{0}$.]

Example A revisited. Given $\vec{a}=4\vec{i}+2\vec{j}-6\vec{k}$ and $\vec{b}=1\vec{i}+3\vec{j}-5\vec{k}$, find a vector perpendicular to $\vec{a}$ and $\vec{b}$.

Consider the parallelogram below.





The area of the parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is given by

$$(\text{base})(\text{height})=\left\|\vec{a}\right\|\left\|\vec{b}\right\|\sin\theta=\left\|\vec{a}\times\vec{b}\right\|.$$

Thus, the area of the triangle with adjacent sides $\vec{a}$ and $\vec{b}$ [half of the parallelogram] is given by
$\frac{1}{2}bh=\frac{1}{2}\left\|\vec{a}\right\|\left\|\vec{b}\right\|\sin\theta=\frac{1}{2}\left\|\vec{a}\times\vec{b}\right\|.$
For some of the text practice exercises, you'll start with points and will need to determine the vectors that are the sides of the parallelogram or triangle.