Calculus 241, section 11.5 Lines in Space

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We have defined vectors as “directed line segments”, which have direction and magnitude.

It will be true that there is a close relationship between vectors and lines: lines also have direction, but lines have a magnitude of “infinity”.

What is the minimum amount of information you need to find the equation of a line in 2-D space?

What is the minimum amount of information you need to find the equation of a line in 3-D space?

We’ll say a vector **L** and line *l* are parallel if the vector **L** is parallel to the vector $\overrightarrow{P_0P}$ which joins two distinct points P_0 and *P* which are on line *l*.

Formally: A line *l* is uniquely determined by a point P_0 on *l* and a vector **L** which is parallel to the line *l*.

In other words, a point $P_0 = (x_0, y_0, z_0)$ is on line *l* iff $\overrightarrow{P_0P}$ is parallel to vector **L**.
But by our definition of “parallel vectors”, we now have,

A point $P_0 = (x_0, y_0, z_0)$ is on line *l* iff $\overrightarrow{P_0P} = t\vec{L}$ for some appropriate scalar *t*.

If $\vec{r}_0 = x_0\,\vec{i} + y_0\,\vec{j} + z_0\vec{k}$, and $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$, then we can write

$$\overrightarrow{P_0P} = \vec{r} - \vec{r}_0 = t\,\vec{L}$$
$$\vec{r} = \vec{r}_0 + t\,\vec{L}$$

This is the **vector equation** of line *l*.

Visual in 3-D: Draw $\vec{L}$; $\vec{r}_0$ takes us to point $P_0 = (x_0, y_0, z_0)$; we then move along in the direction of $\vec{L}$ in increments of *t*.

Example B. Find the vector equation of a line which passes through point (3, 2, 1) and is parallel to $\vec{a} = 4\vec{i} + 2\vec{j} - 6\vec{k}$.

Example B extended. Alternately, we could have written,

These three expressions are called the **parametric equations** of line l.

Formally, given a point $P_0 = (x_0, y_0, z_0)$ and a vector $\vec{L} = a\vec{i} + b\vec{j} + c\vec{k}$ then line l parallel to vector **L** can be written

$$\vec{r} = \vec{r}_0 + t\vec{L} = (x_0 + at)\vec{i} + (y_0 + bt)\vec{j} + (z_0 + ct)\vec{k} \text{ [vector equation]}$$

$$\text{or as } x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \text{ [parametric equations]}$$

If we were to consider a line for which $z = 0$, we'd have the parametric equations for a line in the xy plane discussed back in section 6.7 of the text. [Math 141, Calc II]

But wait, there's more!

As long as a, b, and c are nonzero, we can write the **symmetric equations** of line l.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

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$$\text{or as } \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \text{ [symmetric equations; } a, b, \text{ and } c \text{ are nonzero]}$$

Example B again. Find the symmetric equations of a line which passes through points $P_1 = (3, 2, 1)$ and $P_2 = (7, 4, -5)$.

In Example 4, the text goes the other direction and starts with the symmetric equations, identifies the vector $\vec{L}$ from the denominators, one point by looking in the numerators, and another point by choosing an arbitrary x-value then solving for the necessary y-value and z-value.

For the symmetric equations of line l, we specified, "As long as a, b, and c are nonzero". What if one (or more) does equal zero?

We go back to the parametric equations.

Assume $b = 0$. Then we have $x = x_0 + at,\ \ y = y_0 + 0t,\ \ z = z_0 + ct$, implying that the y-coordinates of all points on the line are constant, and therefore the line is parallel to the xz plane. In this case we'd have

$$y = y_0,\ \ \frac{x - x_0}{a} = \frac{z - z_0}{c}.$$

Last topic for 11.5: Distance from a point to a line. [Theorem 11.12]

Locate a generic point P_1, locate a point on the line such that the line segment connecting the two points is perpendicular to the line, then calculate the distance between the two points.

Or we could take an alternate approach, and think in terms of vectors.

Then, from work done earlier, we'd have $D =$

But, we also have [from Theorem 11.10b],

Example B once more. Find the distance D from the point $P_1 = (4, 5, 6)$ to the line with symmetric equations

$$\frac{x-3}{4} = \frac{y-2}{-2} = \frac{z-1}{-6}.$$