Calculus 241, section 12.1 Vector-Valued Functions Introduction

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In Algebra and Calculus I and II we had real-value functions $y = f(x)$ [rectangular coordinates] and $r = f(\theta)$ [polar coordinates]. In Calculus III (so far) we have encountered $z = f(x, y)$ [multivariable functions], expressed as coordinate points (x, y, z) and as a vector $a\vec{i} + b\vec{j} + c\vec{k}$ [directed line segment from the origin to a point].

Now we introduce a different idea, **vector-valued functions**, $\vec{F}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$.
[See Definition 12.1 in your text.]

		Domain	Range
real-valued functions	$z = f(x, y)$	Real numbers	Real numbers
vector-valued functions	$\vec{F}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$	Real numbers	Vectors

We'll call f_1, f_2 and f_3 the **component functions** of $\vec{F}$, i.e. $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$.
(Think "parametric equations".)

We've already seen a vector-valued function, in section 11.5, although we didn't call it that. A line in 3-D space can be expressed as a function of a real number t:

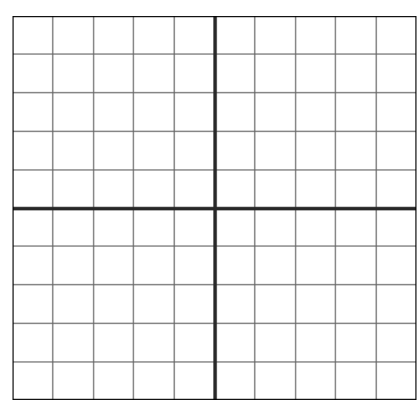
$$\vec{r} = \vec{r}_0 + t\vec{L} = (x_0 + at)\vec{i} + (y_0 + bt)\vec{j} + (z_0 + ct)\vec{k} \text{ [vector equation]}$$

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \text{ [parametric equations].}$$

Example A. Determine the domain of $\vec{F}(t) = e^t\,\vec{i} + \ln(t+2)\,\vec{j} + \sqrt{5-t}\,\vec{k}$.

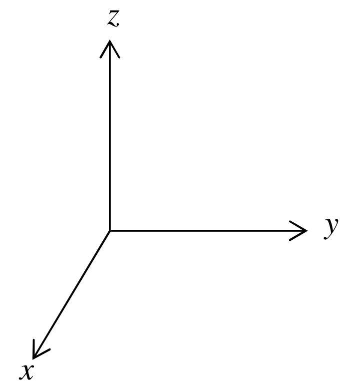
Technically, to graph a vector-valued function $\vec{F}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$ we'd need a 4-D grid to plot points (t, x, y, z). What we *can* do in 3-D space is to sketch only the range of $\vec{F}$, in the form of coordinate points (x, y, z) whose values are determined by the component functions. Additionally, we'll need to think "movement in terms of t": As t increases, the component functions will trace out vectors, so that the resulting curve C has an arrow indicating direction of movement. [Analogy in 2-D: polar graphs]

Example B. Sketch the curve C traced out by $\vec{G}(t) = -2t\,\vec{i} + (1-t)\,\vec{j}$.

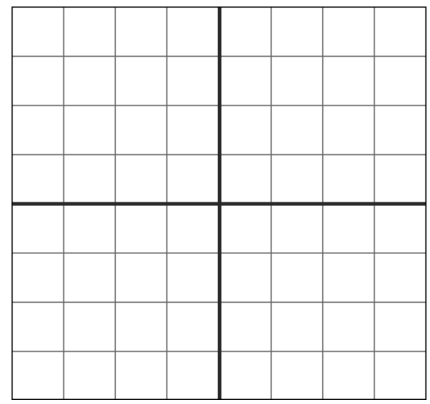


The only remaining question is, "What direction?"

Example C [11.5 Example B]. Sketch the curve C traced out by $\vec{H}(t) = (3+4t)\vec{i} + (2+2t)\vec{j} + (1-6t)\vec{k}$.



Example D. Sketch the curve C traced out by $\vec{F}(t) = 2\cos t\,\vec{j} - 2\sin t\,\vec{k}$.



Your text defines the **standard unit circle** as the curve traced out by the vector-valued function

$$\vec{F}(t) = \cos t\,\vec{i} + \sin t\,\vec{j}.$$

Definition 12.2: Vector-valued functions have the following (not entirely unexpected) properties.

$$(\vec{F} + \vec{G})(t) = \vec{F}(t) + \vec{G}(t) \qquad (\vec{F} - \vec{G})(t) = \vec{F}(t) - \vec{G}(t)$$

$$(f * \vec{F})(t) = f(t) * \vec{F}(t)$$

$$(\vec{F} \bullet \vec{G})(t) = \vec{F}(t) \bullet \vec{G}(t) \qquad (\vec{F} \times \vec{G})(t) = \vec{F}(t) \times \vec{G}(t)$$

$$(\vec{F} \circ g)(t) = \vec{F}(g(t))$$

Example D extended. Sketch the curve C traced out by $\vec{G}(t) = t\,\vec{i} + 2\cos t\,\vec{j} - 2\sin t\,\vec{k}$.

Example E. Sketch the curve C traced out by $\vec{F}(t) = \cos t\,\vec{i} + 2\cos t\,\vec{j} + \sqrt{5}\,\sin t\,\vec{k}$.

Note: Text questions 32 and 33 ask you to find intersections.

32. cylinder $x^2 + y^2 = 4$ and curve traced out by $\vec{F}(t) = t\cos(\pi t)\vec{i} + t\sin(\pi t)\vec{j} + t\,\vec{k}$

33. sphere $x^2 + y^2 + z^2 = 10$ and curve traced out by $\vec{F}(t) = \cos(\pi t)\vec{i} + \sin(\pi t)\vec{j} + t\,\vec{k}$

Suggestions?