Calculus 241, section 12.2 Limits/Continuity & 12.3 Derivatives/Integrals

notes by Tim Pilachowski

What we have so far: **vector-valued functions**, $\vec{F}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$, with a domain of real numbers and a range of vectors.
The functions f_1, f_2 and f_3 are the **component functions** of $\vec{F}$, i.e. $x = f_1(t),\ y = f_2(t),\ z = f_3(t)$.

As was done with regular functions $y = f(x)$ in Calculus I, we take a (short) side trip into a discussion of limits and continuity.

Definition 12.3. "Let **F** [= $\vec{F}$] be a vector-valued function be defined at each point in some open interval containing t_0, except possibly t_0 itself. A vector **L** [= $\vec{L}$] is the **limit of F(*t*) as *t* approaches** t_0 (or **L** is the **limit of F at** t_0) if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |t - t_0| < \delta, \text{ then } \left\| \vec{F}(t) - \vec{L} \right\| < \varepsilon."$$

In this case we write $\lim_{t \to t_0} \vec{F}(t) = \vec{L}$ and say that $\lim_{t \to t_0} \vec{F}(t)$ exists.

In 3-D space, we can visualize an open ball [see Lecture 12.1] or tunnel with center $\vec{L}$ and radius ε. Then $\lim_{t \to t_0} \vec{F}(t) = \vec{L}$ if there is an open interval about t_0 such that $\vec{F}$ assigns to each number in the interval (except possibly t_0) a point in the ball/tunnel.

Most useful to our purposes will be Theorem 12.4.
Let $\vec{F}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$. Then $\vec{F}$ has a limit at t_0 if and only if f_1, f_2 and f_3 have limits at t_0.
That is, $\lim_{t \to t_0} \vec{F}(t) = \left[\lim_{t \to t_0} f_1(t)\right]\vec{i} + \left[\lim_{t \to t_0} f_2(t)\right]\vec{j} + \left[\lim_{t \to t_0} f_3(t)\right]\vec{k}$.
The proof is in the text, so I won't duplicate it here.

Example A. Find $\lim_{t \to -2}\left(e^t\,\vec{i} + \ln(t+2)\,\vec{j} + \sqrt{5-t}\,\vec{k}\right)$ and $\lim_{t \to 5}\left(e^t\,\vec{i} + \ln(t+2)\,\vec{j} + \sqrt{5-t}\,\vec{k}\right)$.

Side note: To follow an example in the text and for one of the practice exercises, you'll need to remember that $\lim_{t \to 0} \frac{\sin t}{t} = 1$.

Only a couple more theoretical things are needed to finish up section 12.2.

First of all, properties.

Theorem 12.5. Let $\vec{F}(t)$ and $\vec{G}(t)$ be vector-valued functions, and $f(t)$ and $g(t)$ be real-valued functions, for which all limits exist and particularly $\lim_{s\to s_0} g(s) = t_0$.

$$\lim_{t\to t_0}\left(\vec{F}+\vec{G}\right)(t) = \lim_{t\to t_0}\vec{F}(t) + \lim_{t\to t_0}\vec{G}(t) \qquad \lim_{t\to t_0}\left(\vec{F}-\vec{G}\right)(t) = \lim_{t\to t_0}\vec{F}(t) - \lim_{t\to t_0}\vec{G}(t)$$

$$\lim_{t\to t_0}\left(f * \vec{F}\right)(t) = \lim_{t\to t_0} f(t) * \lim_{t\to t_0}\vec{F}(t)$$

$$\lim_{t\to t_0}\left(\vec{F}\bullet\vec{G}\right)(t) = \lim_{t\to t_0}\vec{F}(t) \bullet \lim_{t\to t_0}\vec{G}(t) \qquad \lim_{t\to t_0}\left(\vec{F}\times\vec{G}\right)(t) = \lim_{t\to t_0}\vec{F}(t) \times \lim_{t\to t_0}\vec{G}(t)$$

$\lim_{s\to s_0}\left(\vec{F}\circ g\right)(s) = \lim_{t\to t_0}\vec{F}(t)$ if $g(s) \neq t_0$ for all s in an open interval about s_0

Proofs are in the text, so I won't duplicate them here.

Second, continuity.

Definition 12.6. "A vector-valued function $\vec{F}$ is **continuous** at a point t_0 in its domain if $\lim_{t\to t_0}\vec{F}(t) = \vec{F}(t_0)$.

Theorem 12.7. "A vector-valued function $\vec{F}$ is continuous at t_0 if and only if each of its component functions is continuous at t_0.

<u>Here begins section 12.3.</u>

Definition 12.8. "Let t_0 be a number in the domain of a vector-valued function **F** [= $\vec{F}$]. If $\lim_{t\to t_0}\frac{\vec{F}(t)-\vec{F}(t_0)}{t-t_0}$ exists, we call this limit the **derivative** of **F** at t_0 and write $\vec{F}'(t_0) = \lim_{t\to t_0}\frac{\vec{F}(t)-\vec{F}(t_0)}{t-t_0}$."

We'll also use the Leibnitz notation, $\vec{F}'(t) = \frac{d\vec{F}}{dt}$.

Informally stated, just as the derivative of $y = f(x)$ was derived as the limit of slopes of secant lines providing the slope of the tangent to the curve, the derivative $\vec{F}'(t)$ provides us with a vector which is tangent to the curve C which is traced out by a vector-valued function $\vec{F}(t)$.

Theorem 12.9. "Let $\vec{F}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$. Then $\vec{F}$ is differentiable at t_0 if and only if f_1, f_2 and f_3 are differentiable at t_0. In that case, $\vec{F}'(t) = f_1'(t)\vec{i} + f_2'(t)\vec{j} + f_3'(t)\vec{k}$."

Example B. Let $\vec{H}(t) = (3+4t)\vec{i} + (2+2t)\vec{j} + (1-6t)\vec{k}$. Find $\vec{H}'(t)$.

As illustrated in Example B, the derivative of a linear vector-valued function $r_0 + t\vec{L}$ is a constant vector-valued function parallel to $\vec{L}$.

Example C. Let $\vec{G}(t) = t^2\,\vec{i} + 2\cos t\,\vec{j} - 2t\sin t\,\vec{k}$. Find $\vec{G}'\left(\frac{\pi}{2}\right)$.

Note that the product rule was needed for the *z*-component.

Almost all of the differentiation rules from Calculus I have counterparts for vector-valued functions.
Theorem 12.10. "Let $\vec{F}$, $\vec{G}$, and *f* be differentiable at t_0, and let *g* be differentiable at s_0 with $g(s_0) = t_0$.

$$\left(\vec{F}' + \vec{G}'\right)(t) = \vec{F}'(t) + \vec{G}'(t) \qquad \left(\vec{F}' - \vec{G}'\right)(t) = \vec{F}'(t) - \vec{G}'(t)$$

$$\left(f * \vec{F}\right)'(t) = f(t) * \vec{F}'(t) + f'(t) * \vec{F}(t)$$

$$\left(\vec{F} \bullet \vec{G}\right)'(t) = \vec{F}'(t) \bullet \vec{G}(t) + \vec{F}(t) \bullet \vec{G}'(t) \qquad \left(\vec{F} \times \vec{G}\right)'(t) = \vec{F}'(t) \times \vec{G}(t) + \vec{F}(t) \times \vec{G}'(t)$$

$$\left(\vec{F} \circ g\right)'(s_0) = \vec{F}'[g(s_0)] * g'(s_0) = \vec{F}'(t_0) * g'(s_0)$$

When there is a choice of methods, I recommend choosing the simpler one.

Example D. Let $\vec{F}(t) = t\,\vec{i} + \ln t\,\vec{j} + 5\vec{k}$ and $\vec{G}(t) = t\,\vec{i} + 2\cos t\,\vec{j}$. Find $\left(\vec{F} \times \vec{G}\right)'(t)$.
If we use the rule above, the right-hand side would mean doing two cross products.
In this case, it may be easier to do one cross product on the left, then differentiate.

Possibly the most useful of the properties above will be the last one, the "chain rule".

Corollary 12.11. “Let $\vec{F}$ be differentiable on an interval *I*, and assume there is a number *c* such that $\left\| \vec{F}(t) \right\| = c$ for *t* in *I*. Then $\vec{F}(t) \bullet \vec{F}'(t) = 0$ for *t* in *I*.

The text has a three-line proof.
More important for our purposes are the implications of Corollary 12.11.
If $\left\| \vec{F}(t) \right\|$ is constant, then for each *t* in the domain of $\vec{F}$ one of the following is true.

$$\vec{F}(t) = \vec{0}$$

$$\vec{F}'(t) = \vec{0} \text{ (e.g., if } \vec{F}(t) = a\vec{i} + b\vec{j} + c\vec{k}\text{)}$$

$$\vec{F}(t) \text{ and } \vec{F}'(t) \text{ are perpendicular (e.g. if } \vec{F}(t) \text{ traces out a circle)}$$

Lecture 12.1 Example E revisited. Given $\vec{F}(t) = \cos t\,\vec{i} + 2\cos t\,\vec{j} + \sqrt{5}\sin t\,\vec{k}$, first find $\vec{F}'(t)$, then show that $\vec{F}(t)$ and $\vec{F}'(t)$ are perpendicular for all values of *t* in the domain.
(Do this one on your own, for practice.)

The second derivative of a vector-valued function is defined as the derivative of the first derivative of a vector valued function.

$$\vec{F}(t) = f_1\,\vec{i} + f_2\,\vec{j} + f_3\,\vec{k}$$

$$\vec{F}'(t) = f_1'\,\vec{i} + f_2'\,\vec{j} + f_3'\,\vec{k}$$

$$\vec{F}''(t) = f_1''\,\vec{i} + f_2''\,\vec{j} + f_3''\,\vec{k}$$

Example C revisited. Let $\vec{G}(t) = t^2\,\vec{i} + 2\cos t\,\vec{j} - 2t\sin t\,\vec{k}$. Find $\vec{G}''(t)$.

When we apply vector-valued functions to applications involving motion with respect to time, we get results similar to those found in Calculus I.

Position: $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ (**radial** or **radius** vector)

$\vec{r}(t) - \vec{r}(t_0)$ (**displacement** vector, from initial position to current one)

$$\text{Velocity: } \vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

$$\text{Speed: } \|\vec{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$\text{Acceleration: } \vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}$$

Example E. Find the position, velocity and speed of an object having acceleration $\vec{a}(t) = -32\vec{k}$, initial velocity $\vec{v}_0 = 2\vec{i} + 2\vec{j} + \vec{k}$, and initial position $\vec{r}_0 = 3\vec{i} + 3\vec{k}$.

For the example above, we made use of Theorem 12.12. Let $\vec{F}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$ where f_1, f_2 and f_3 are continuous on [a, b].

$$\int \vec{F}(t)\,dt = \left(\int f_1(t)\,dt\right)\vec{i} + \left(\int f_2(t)\,dt\right)\vec{j} + \left(\int f_3(t)\,dt\right)\vec{k}$$

$$\int_a^b \vec{F}(t)\,dt = \left(\int_a^b f_1(t)\,dt\right)\vec{i} + \left(\int_a^b f_2(t)\,dt\right)\vec{j} + \left(\int_a^b f_3(t)\,dt\right)\vec{k}$$

For assistance with this same idea applied to applications involving objects subject only to the gravity of earth (text exercises 47-49), see the text Example 10 and Example 11.