Calculus 241, section 12.4 Space Curves & Lengths

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In section 12.1, we discussed sketching the range of a vector-valued function, $\vec{F}$, in the form of coordinate points (*x, y, z*) whose values are determined by the endpoints of the component functions. The underlying concept was "movement in terms of *t*": as *t* increases, the component functions will trace out vectors, so that the resulting curve *C* has an arrow indicating direction of movement. Beginning in this section we'll be a little more formal and restricted when talking about curves.

Definition 12.13. "A **space curve** (or simply **curve**) is the range of a continuous vector-valued function on an interval of real numbers."

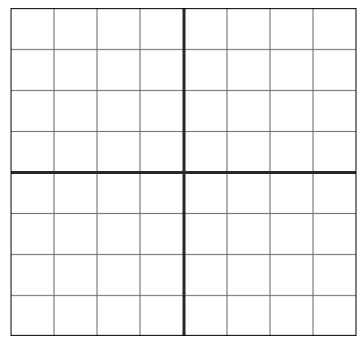
Given a vector-valued function **r** = $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, we'll say that a curve *C* is **parametrized** by $\vec{r}$, or that $\vec{r}$ is a **parametrization** of *C*. As a shortcut, the text (and we) will at times refer to "the curve $\vec{r}$".

This definition of a curve in space includes all of the graphs of continuous real-valued functions encountered in Algebra and Calculus I and II. A continuous function $y = f(x)$ can be parametrized by

$$x = t,\ y = f(t), \text{ that is } \vec{r}(t) = t\vec{i} + f(t)\vec{j}.$$

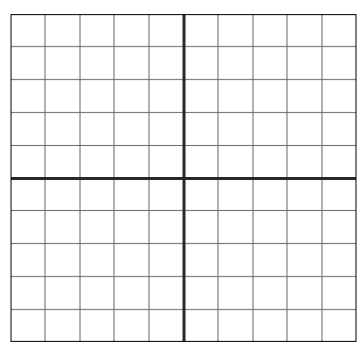
Definition 12.14. "A curve *C* is **closed** if it has a parametrization whose domain is a closed interval [*a*, *b*] such that $\vec{r}(a) = \vec{r}(b)$, but otherwise $\vec{r}(t_1) \neq \vec{r}(t_2)$ for $t_1 \neq t_2$, with at most finitely many exceptions."

Example A. Show that the curve *C* parametrized by $\vec{r}(t) = 2\cos t\,\vec{j} - 2\sin t\,\vec{k}$ is closed.

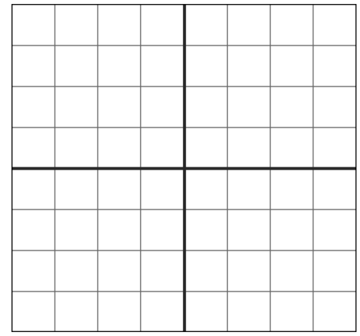


Examples B: Determine whether or not the following curves are closed.

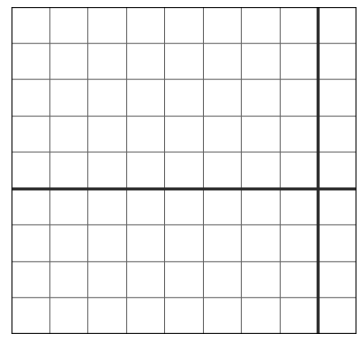
a) $x = 2\cos t(1-\cos t),\ y = 2\sin t(1-\cos t)$ on the interval [0, 2π].



b) $x = 4\cos(2t)\cos(t)$, $y = 4\cos(2t)\sin(t)$ on the interval $[0, 2\pi]$.



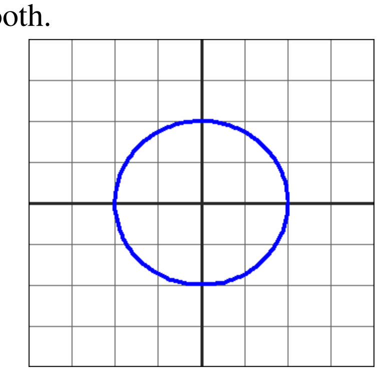
c) $x = t^3 - 3t^2$, $y = t^3 - 3t$ on any closed interval



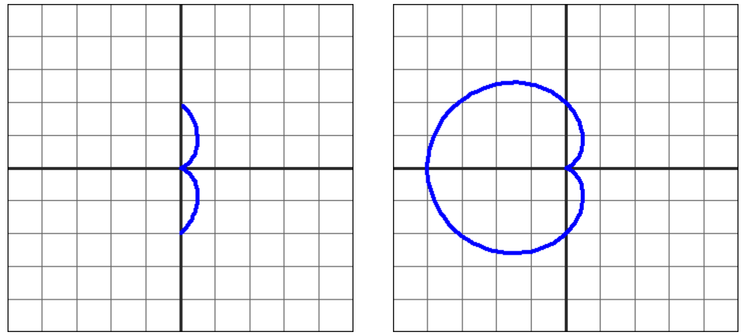
The examples above are in the 2-D plane, but the concepts extend to curves in 3-D space.
The circular helix $\vec{G}(t) = t\,\vec{i} + 2\cos t\,\vec{j} - 2\sin t\,\vec{k}$ is not closed on any interval in its domain.
The *x*-component will be different for every value of *t* in the domain.

Definition 12.15a. "A vector-valued function $\vec{r}$ defined on an interval *I* is **smooth** if $\vec{r}$ has a continuous derivative on *I* and $\vec{r}\,'(t) \neq \vec{0}$ for each interior point *t*. A curve *C* is **smooth** if it has a smooth parametrization."
Definition 12.15b. "A continuous vector-valued function $\vec{r}$ defined on an interval *I* is **piecewise smooth** if *I* is composed of a finite number of subintervals on each of which $\vec{r}$ is smooth. A curve *C* is **piecewise smooth** if it has a piecewise smooth parametrization."

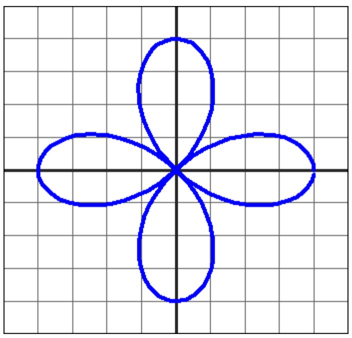
Example A revisited. The curve *C* parametrized by $\vec{r}(t) = 2\cos t\,\vec{j} - 2\sin t\,\vec{k}$ is smooth.



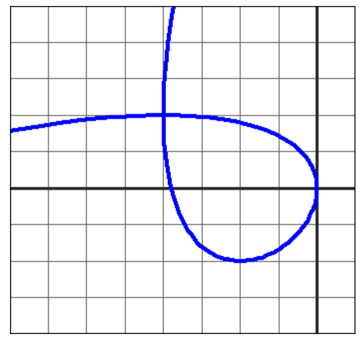
Examples B revisited: a) $\vec{r}(t) = 2\cos t(1-\cos t)\vec{i} + 2\sin t(1-\cos t)\vec{j}$ is not smooth across its entire domain.



b) $x = 4\cos(2t)\cos(t),\ y = 4\cos(2t)\sin(t)$ is smooth



c) $x = t^3 - 3t^2,\ y = t^3 - 3t$ is smooth



The circular helix $\vec{G}(t) = t\vec{i} + 2\cos t\,\vec{j} - 2\sin t\,\vec{k}$ is smooth.

In Example 5 (and the explanation which precedes it), the text shows that lines and line segments in 3-D space are both continuous and smooth.

Last topic that you'll need for section 12.4: Length of a curve in 3-D space.

Do you remember the formula for length of a curve derived in section 6.2?

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

Do you remember the formula for length of a parametrized curve derived in section 6.8?

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

In a fashion similar to that followed in those sections, the text develops a formula for the length of a curve in 3-D space.

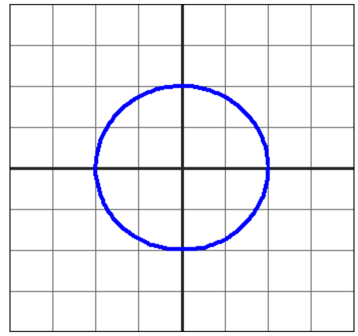
Definition 12.16. Let C be a curve with a piecewise smooth parametrization $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ defined on $[a, b]$. Then the length L of C is given by

$$L = \int_a^b \|\vec{r}'(t)\|\,dt = \int_a^b \left\|\frac{d\vec{r}}{dt}\right\|dt$$

$$= \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

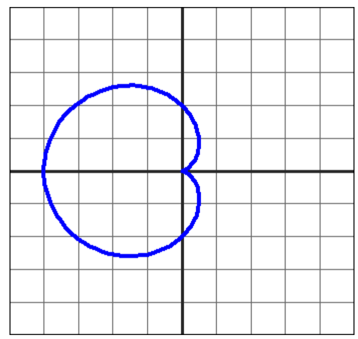
$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Example A once again. Find the length L on the interval [0, 2π] of the curve C parametrized by $\vec{r}(t) = 2\cos t\,\vec{j} - 2\sin t\,\vec{k}$.

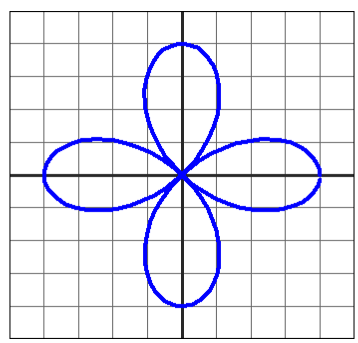


Note that we get the circumference of a circle with radius 2.
Important note: We had to be careful, and avoid a parametrization and interval that traced out the curve C more than once.

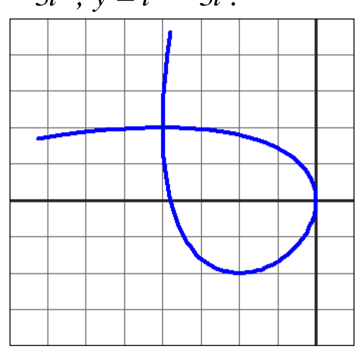
Examples B once again: a) The length of $\vec{r}(t) = 2\cos t(1-\cos t)\vec{i} + 2\sin t(1-\cos t)\vec{j}$ on the interval [0, 2π] would require integration by substitution.



b) The length of $\vec{r}(t) = 4\cos(2t)\cos(t)\vec{i} + 4\cos(2t)\sin(t)\vec{j}$ on the interval [0, 2π] would require use of the double-angle identities, along with trigonometric methods from section 8.2.



c) Find the length L on the interval [– 1.3, 2.3] of the curve C parametrized by $x = t^3 - 3t^2,\ y = t^3 - 3t$.



Find the length L on the interval $[0, 6\pi]$ of the circular helix $\vec{G}(t) = t\,\vec{i} + 2\cos t\,\vec{j} - 2\sin t\,\vec{k}$.

Two final notes:
The length of a line segment in 3-D space derives to the same formula as the length of the vector connecting the two points.
Every curve has many parametrizations. Since we would get the same length for any piecewise smooth curve by using any of its parametrizations, we can say that the length of a curve is **independent of parametrization**.