Calculus 241, section 12.5-12.6 Tangent & Normal to Curves & Curvature

notes by Tim Pilachowski

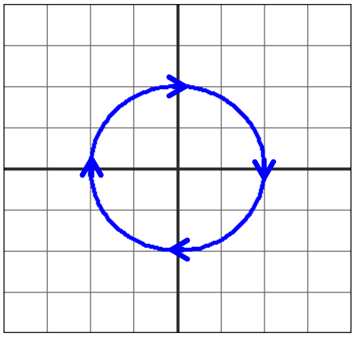
Definition 12.17. "Let C be a smooth curve and **r** [$=\vec{r}$] a smooth parametrization of C defined on an interval I. Then for any interior point t of I, the **tangent vector T**(t) [$=\vec{T}(t)$] at the point $\vec{r}(t)$ is defined by

$$\vec{T}(t)=\frac{\vec{r}'(t)}{\left\|\vec{r}'(t)\right\|}=\frac{d\vec{r}/dt}{\left\|d\vec{r}/dt\right\|}."$$

Note that $\vec{T}(t)$ is a unit vector in the same direction as $\vec{r}'(t)$.

Example A. Given the curve C parametrized by $\vec{r}(t)=2\cos t\,\vec{j}-2\sin t\,\vec{k}$, find a formula for the tangent vector $\vec{T}(t)$ and then evaluate $\vec{T}\left(\frac{5\pi}{4}\right)$.

We graphed this curve in Lecture 12.1 Example D, and saw a circle in the yz plane centered at (0, 0, 0) with a radius of 2 and a clockwise movement.



Given the shape traced out by $\vec{r}(t)$ in this case, it's not really surprising that $\vec{T}(t)\bullet\vec{r}(t)=0$ for all values of t, i.e. that the position vector and the tangent vector are perpendicular for all t.

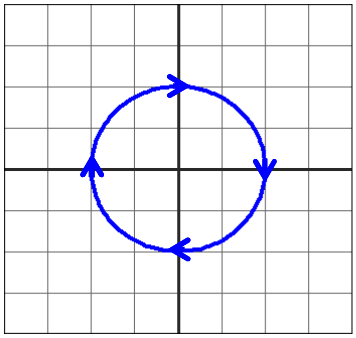
Example B. Given the circular helix parametrized by $\vec{r}(t)=t\,\vec{i}+2\cos t\,\vec{j}-2\sin t\,\vec{k}$, find a formula for the tangent vector $\vec{T}(t)$.

Definition 12.18. "Let C be a smooth curve and **r** [$=\vec{r}$] a smooth parametrization of C defined on an interval I such that $\vec{r}'$ is smooth. Then for any interior point t of I for which $\vec{T}'(t)\neq\vec{0}$, the **normal vector N**(t) [$=\vec{N}(t)$] at the point $\vec{r}(t)$ is defined by

$$\vec{N}(t)=\frac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|}=\frac{d\vec{T}/dt}{\left\|d\vec{T}/dt\right\|}."$$

Note that $\vec{N}(t)$ is a unit vector in the same direction as $T'(t)$.

Example A revisited. Given the curve C parametrized by $\vec{r}(t) = 2\cos t\,\vec{j} - 2\sin t\,\vec{k}$, find a formula for the normal vector $\vec{N}(t)$ and then evaluate $\vec{N}\left(\frac{5\pi}{4}\right)$.



Note that the normal to a circle points toward the center of the circle, opposite the direction of the position vector $\vec{r}(t)$, and is perpendicular to the tangent vector $\vec{T}(t)$ for all values of t.

Example B revisited. Given the circular helix parametrized by $\vec{r}(t) = t\,\vec{i} + 2\cos t\,\vec{j} - 2\sin t\,\vec{k}$, find a formula for the normal vector $\vec{N}(t)$.

The normal vectors to this circular helix are always perpendicular to, and pointed toward, the x-axis.

Here begins section 12.6.

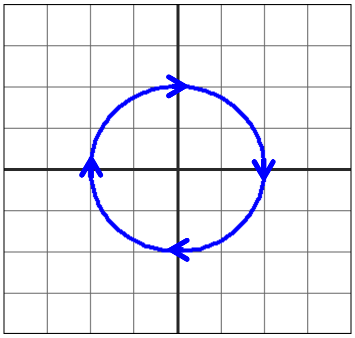
We begin with a curve C. While the length of the tangent vector is always 1, the change in direction from one tangent vector to the next can vary from "not at all" if C is a line, to "a whole lot" if C has, for example, a corkscrew shape. In other words, the rate of change of the tangent vector with respect to the arc length function s is related to the rate at which C bends in space. The text uses the Chain Rule to derive a formula for the curvature κ of the curve C.

Definition 12.19. "Let C have a smooth parametrization **r** [$= \vec{r}$] such that $\vec{r}\,'$ is differentiable. Then the curvature κ of C is defined by the formula

$$\kappa(t) = \frac{\left\|\vec{T}\,'(t)\right\|}{\left\|\vec{r}\,'(t)\right\|} = \frac{\left\| d\vec{T}/dt \right\|}{\left\| d\vec{r}/dt \right\|}."$$

Note that, by definition, $\kappa(t)$ must always be 0 or a positive number.

Example A once again. Given the curve C parametrized by $\vec{r}(t) = 2\cos t\,\vec{j} - 2\sin t\,\vec{k}$, find a formula for the curvature $\kappa(t)$. *answer*: 1/2



The text, in Example 1, demonstrates that the curvature of a circle of radius r is $\frac{1}{r}$.

Example B once again. Given the circular helix parametrized by $\vec{r}(t) = t\,\vec{i} + 2\cos t\,\vec{j} - 2\sin t\,\vec{k}$, find a formula for the curvature $\kappa(t)$.

I'll leave it to you to show that the curvature of this circular helix is constant. *answer*: $\frac{2}{\sqrt{5}}$

The text develops an alternative formula that, in some cases, will make the computations for curvature easier.

Given that $\vec{v}(t) = \vec{r}\,'(t)$, and $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$, then $\kappa(t) = \dfrac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|^3}$.

Example C [text practice exercise 4]. Given the curve parametrized by
$\vec{r}(t) = \frac{1}{3}(1+t)^{3/2}\,\vec{i} + \frac{1}{3}(1-t)^{3/2}\,\vec{j} + \frac{\sqrt{2}}{2}t\,\vec{k}$, find a formula for the curvature $\kappa(t)$.

You should get

$$\vec{v}(t) = \vec{r}\,'(t) = \frac{1}{2}(1+t)^{1/2}\,\vec{i} - \frac{1}{2}(1-t)^{1/2}\,\vec{j} + \frac{\sqrt{2}}{2}\vec{k}$$

$$a(t) = \vec{r}\,''(t) = \frac{1}{4}(1+t)^{-1/2}\,\vec{i} + \frac{1}{4}(1-t)^{-1/2}\,\vec{j}$$

$$\vec{v} \times \vec{a} = -\frac{\sqrt{2}}{8\sqrt{1-t}}\vec{i} + \frac{\sqrt{2}}{8\sqrt{1+t}}\vec{j} + \frac{1}{4\sqrt{1-t^2}}\vec{k}$$

$$\kappa(t) = \frac{1}{4}\sqrt{\frac{2}{1-t^2}}$$