Calculus 241, section 13.1 Functions of Several Variables
13.2 Limits and Continuity

notes by Tim Pilachowski

In Algebra and in Calculus I and II, the functions you dealt with have mostly been functions of one variable, like $f(x) = x^2 - x + 1$. Pick an x, plug it in, and calculate. But even then, you worked with functions of more than one variable, even if they were not called that at the time.

Defnition 13.1. “A **function of several variables** consists of two parts: a **domain**, which is a collection of points in the plane or in space, and a **rule**, which assigns to each member of the domain one and only one real number.”

Contrast this with a vector-valued function, whose range consists of vectors.

Consider the basic formulas from geometry. Some, like those for circles, are functions of one variable. We could write area of a circle as a function of radius: $A(r) = \pi r^2$. The rectangle formulas require two variables, one for each dimension. If we let x = length and y = width, then the perimeter formula is $P(x, y) = 2x + 2y$. The area formula would be $A(x, y) = xy$. Note that the domain of each of these two rectangle functions is $x \geq 0$, $y \geq 0$.

Example A: Exponential growth and decay (section 2.3) could have been written in function notation: $y(y_0, k, t) = y_0 e^{kt}$. (Note that e is not a variable but the number called Euler’s number.) One, two or all three variables may change from one scenario to the next:

y_0	k	t	$y(y_0, k, t) = y_0 e^{kt}$ exact value	approximate value

The domain of the exponential growth function in this application is $y_0 \geq 0$, $k \geq 0$, $t \geq 0$.

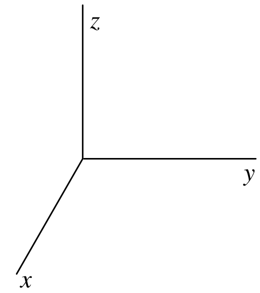
The text notes four more functions, in addition to “area of a rectangle” that will be encountered later in chapter 13, each of which is given a physical interpretation.

2. $f(x, y, z) = xyz$ for $x \geq 0$, $y \geq 0$, $z \geq 0$: volume of a rectangular parallelpiped (box)
3. $f(x, y, z) = 2xy + 2yz + 2xz$ for $x \geq 0$, $y \geq 0$, $z \geq 0$: surface area of a rectangular parallelpiped (box)
4. $f(x, y, z) = \dfrac{c}{x^2 + y^2 + z^2}$ where $c > 0$ is a constant: magnitude of the gravitational force exerted by the sun, located at (0, 0, 0) , on a unit mass located at (x, y, z).
5. $f(x, y, z) = \dfrac{c}{\sqrt{x^2 + y^2}}$ where c is a constant: strength of the electric field at (x, y, z) due to an infinitely long wire lying along the z-axis.

All of the usual arithmetic combinations of functions work the same way for functions of several variables ($f + g, f - g, fg$) as well as the usual restrictions applied to division (the domain of $\dfrac{f}{g}$ is limited to values for which $g \neq 0$). Polynomial and rational functions of several variables are defined in a manner similar to that of functions of one variable. Composition of functions is analogous, as well: Given $g(x)$ and $f(x, y)$, $(g \circ f)(x, y) = g(f(x, y))$.

A function of two variables, $z = f(x, y)$, can be graphed on a three-dimensional grid. Picture the corner of a room where the length and width of the floor form the x-axis and y-axis. The vertical where the two walls meet is the z-axis. Where the Cartesian 2-D grid is split into 4 quadrants, our 3-D grid is divided into 8 octants.

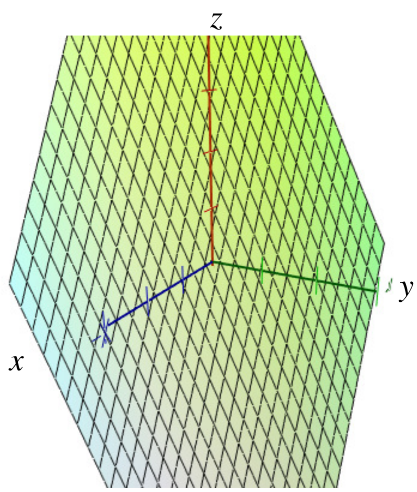
Example B: Graph the first-octant portion of the plane $f(x, y) = 4 - \frac{4}{3}x - \frac{2}{3}y$.

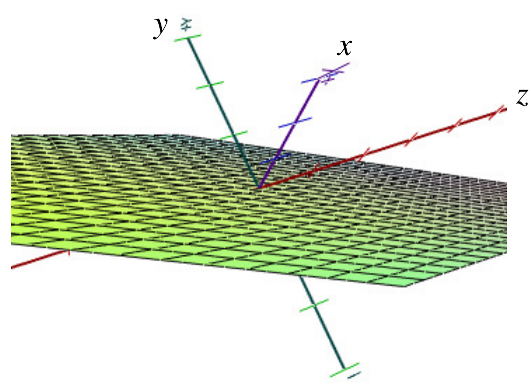


The "standard" orientation, while helpful from a conceptual view, will not always provide a clear picture of the function. A three-dimensional shape translated into two dimensions can obscure the true nature of the shape. It would be nice to be able to rotate the picture through the three dimensions to get a clearer idea.

A very good, and free, graphing calculator for your computer can be downloaded from www.graphcalc.com. (It might be useful, particularly if you don't own your own copy of Matlab.) This utility will allow you to graph both two- and three-dimensional graphs, and rotate the three-dimensional graphs either automatically or manually. (It's the utility I used for the examples below.) It also has the ability to do some evaluations, as well as to graph parametric and polar equations. (You have your choice of colors to draw the graph in, too.)

Example C: Graph and explore the rectangle perimeter function $f(x, y) = 2x + 2y$.
The graph shown to the left below has been drawn on the axes as oriented above, but it's hard to tell what shape it really has. The graph to the right has been rotated to illustrate more clearly that the shape is a flat surface—in geometry terms a plane.

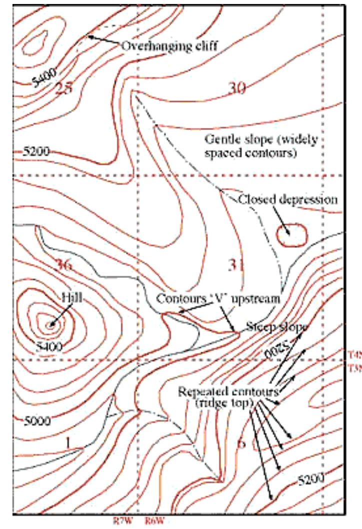




Let's go back briefly to the type of problems we've encountered before this. Even when we needed to use a function of more than one variable, there was enough information given so that we could do some sort of substitution and turn it into an equation with one variable that we could solve.

Example D: Farmer Bob has a rectangular corral and 120 feet of fencing. Write an equation in one variable that represents the area of the corral. *Answer*: $A = x(60 - x)$

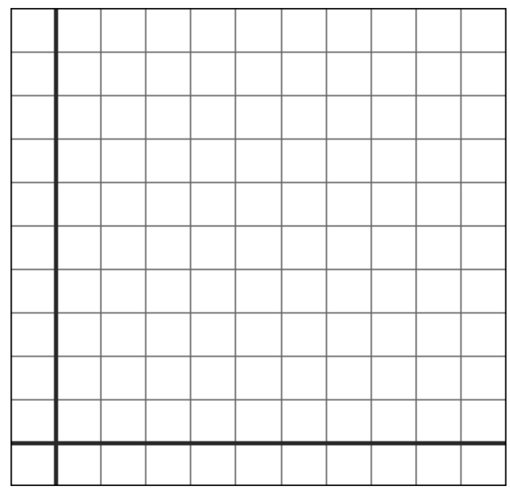
A similar process provides a means of taking a three-dimensional object and considering it in two dimensions. The concept is the same one used in drawing topographical maps that show the elevations of terrain. The flat surface is laid out in latitude and longitude (the x and y) while a series of curves show the elevation (z) of the terrain. Tracing along a curve shows the places on the ground that all have the same elevation. Widely spaced curves indicate a gentler slope. Curves close together indicate a steep slope.



If we choose a series of values for z, the result is a series of equations in x and y that can be graphed as usual on the Cartesian grid. These are called *level curves*. Your text exercises ask you to graph the first octant. Some of my examples use the whole Cartesian grid to give you a more comprehensive picture.

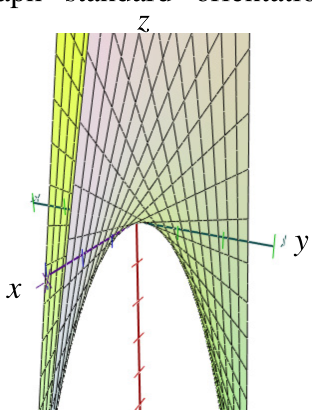
Example C completed: Draw level curves for $f(x, y) = 2x + 2y$ for $z = 10, 8, 6, 4$, and 2.
Given $z = 10$, then $10 = 2x + 2y \;\Rightarrow\; 10 - 2x = 2y \;\Rightarrow\; 5 - x = y$.
The others are found in a similar fashion.

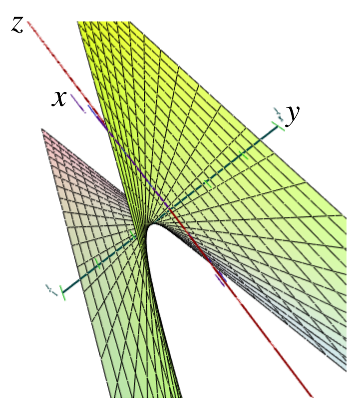
z	$z = 2x + 2y$	$f(x) = y = \ldots$
10	$10 = 2x + 2y$	$y = 5 - x$



Each level curve on the grid represents a different value for z.
The flat surface of $z = 2x + 2y$ rises at an angle as z increases.

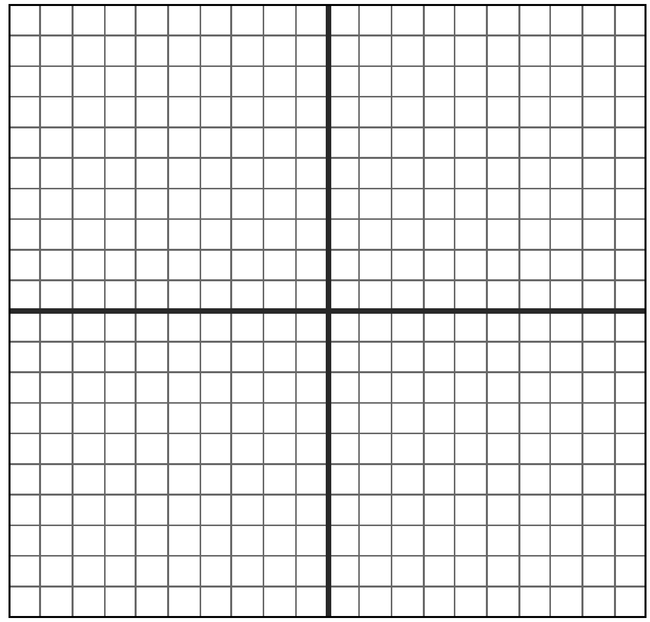
Example E: Graph and explore the rectangle area function $f(x, y) = xy$. Sketch level curves for $z = 10, 8, 6, 4, 2$.
The graph "standard" orientation of the x-y-z axes is on the left, and a rotation of the graph on the right.



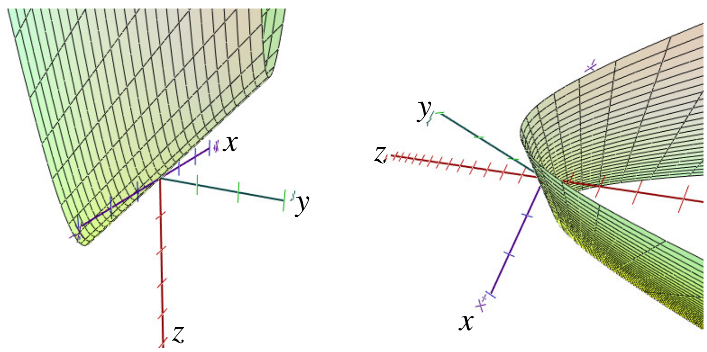


The "twist" (called a "saddle point") in the 3-D surface is easier to see in the rotated version. The symmetric nature of the twist centered at the origin is even more apparent from the level curves.

z	$z = xy$	$f(x) = y = \ldots$

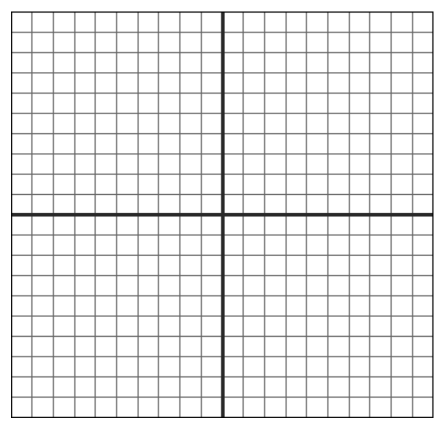


Example F: Draw the 3-D graph and level curves for the function $f(x, y) = x^2 - y$.



The orientation on the left shows a hint of the parabolic nature of the surface, which becomes clearer in the rotation. The level curves make it obvious.

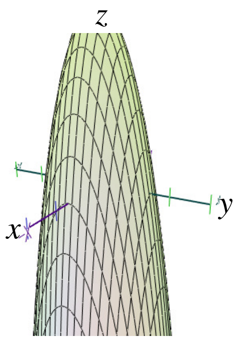
z	$z = x^2 - y$	$f(x) = y = \ldots$

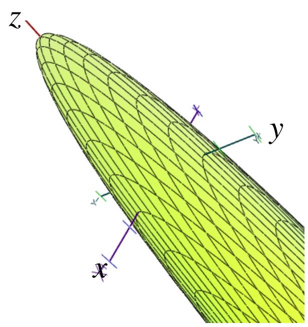


Something we'll be interested in later on is developing a mathematical process to determine the location and nature of the minimum of the shape.

Example F extended: Draw the 3-D graph and level curves for the function $f(x, y) = x^2$.

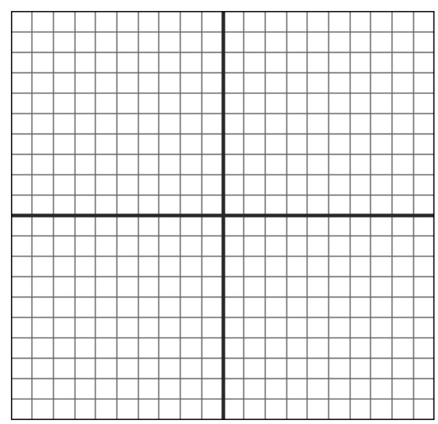
Example G: Draw the 3-D graph and level curves for the function $f(x, y) = 9 - x^2 - y^2$.





The surface on the left shows something of the curve, but the top is outside the viewing window. Tilting the graph slightly allows us to see the maximum. The level curves show the circular nature of the shape. At a later time we'll be developing a method of finding the location of that absolute maximum.

z	$z = 9 - x^2 - y^2$	equation in x and y



Example G extended: Draw the 3-D graph and level curves for the function $f(x, y) = \sqrt{9 - x^2 - y^2}$

(See text Example 4.)

See the text, following Example 5, for some very good graphics of surfaces which are the graphs of various functions $f(x, y)$.

Moving to functions $f(x, y, z)$, we have the same problem in graphing that we did in graphing vector-valued functions: the need for a four-dimensional grid. For vector-valued functions, we chose to sketch only the curve C traced out by the endpoints of the vectors. In a similar fashion, for functions $f(x, y, z)$ we will sketch the **level surfaces**, the surface formed from the set of points for which $f(x, y, z)$ equals a chosen constant c.

Most of the level surfaces you'll be asked to sketch will be similar to the surfaces $z = f(x, y)$ that we've already seen. Level surfaces encountered in Chapter 11 were spheres, cylinders and planes. The Examples done above provide a few other possibilities.

The text explores various **quadric surfaces**, three-dimensional version of the conic sections of chapter 10. Read through the text and look at the graphics to familiarize yourself with these.

All we're going to note about section 13.2 is that the notions of limits and continuity explored earlier carry over to functions of several variables.