Calculus 241, section 13.3 Partial Derivatives

notes by Tim Pilachowski

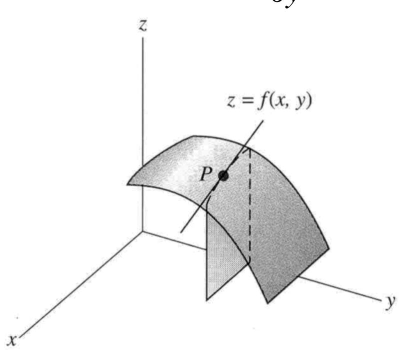
Example A: Given $f(x, y) = 9 - x^2 - y^2$, find $\dfrac{f(x+h, y) - f(x, y)}{h}$.

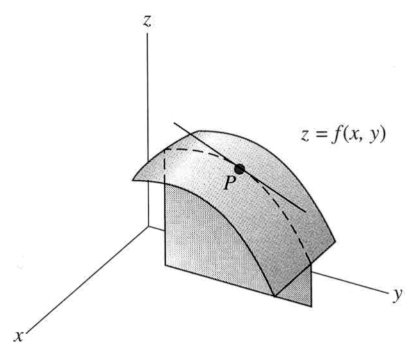
When working with functions of more than one variable, the question in calculus becomes: how can we evaluate the rate of change? The answer is called a *partial derivative*. Given a function $f(x, y)$ or $f(x, y, z)$, the partial derivative of f with respect to x, $\dfrac{\partial f}{\partial x} = f_x$, is found by treating all variables other than x as constants.

Technically, we're finding $\displaystyle\lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$, $\displaystyle\lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$, etc. [Definition 13.4 and formula (2)] The partial derivatives $\dfrac{\partial f}{\partial y} = f_y$ and $\dfrac{\partial f}{\partial z} = f_z$ have analogous definitions.

Example A extended: Given the function $f(x, y) = 9 - x^2 - y^2$ find $\dfrac{\partial f}{\partial x} = f_x$ and $\dfrac{\partial f}{\partial y} = f_y$. *answers*: $-2x$; $-2y$

A geometric interpretation of partial derivative is pictured below. In each figure $f(x, y)$ is the curved surface. In the figure on the left, with y treated as a constant, the tangent line goes the same general direction as the x-axis, and $\dfrac{\partial f}{\partial x}$ is the slope of that tangent at point P. With x treated as a constant, the tangent line goes the same general direction as the y-axis, and $\dfrac{\partial f}{\partial y}$ is the slope of that tangent at point P.





The text gives a more formal explanation, connecting $\dfrac{\partial f}{\partial x} = f_x$ to a tangent vector $\vec{i} + f_x(x_0, y_0)\vec{k}$, and connecting $\dfrac{\partial f}{\partial y} = f_y$ to a tangent vector $\vec{j} + f_y(x_0, y_0)\vec{k}$.

All of the usual derivative "rules" will apply: constant multiple rule, sum rule, power rule, exponential rule, product rule, chain rule.(See the text for the full list.)

Example B: Given the function $f(x, y, z) = x^2yz - ze^{xy} + \frac{x}{y}\ln(z)$ find $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$ and $\frac{\partial f}{\partial z} = f_z$.

answers: $2xyz - yze^{xy} + \frac{1}{y}\ln(z)$; $x^2z - xze^{xy} - \frac{x}{y^2}\ln(z)$; $x^2y - e^{xy} + \frac{x}{yz}$

Example C: For $f(x, y) = x^2 + 3xy + y + 7$, determine $\frac{\partial f}{\partial x}(5, 8) = f_x(5, 8)$ and $\frac{\partial f}{\partial y}(5, 8) = f_y(5, 8)$.

answers: 34; 16

Example D: Given $z = x^2 + 4xy + y^2 - 12y$, find values of x, y and z such that both $z_x = 0$ and $z_y = 0$.

answer: $(x, y, z) = (4, -2, 12)$

Example E: For $f(x, y) = \ln|2x+3y| + \sin(xy)$ determine $\frac{\partial^2 f}{\partial x^2} = f_{xx}$, $\frac{\partial^2 f}{\partial y^2} = f_{yy}$, $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$, $\frac{\partial^2 f}{\partial y \partial x} = f_{xy}$.

Just like regular derivatives, higher order partial derivatives can be found. The first two are called respectively the second partial derivative with respect to *x* and the second partial derivative with respect to *y*. The second two are often called "mixed partial derivatives". In this case, as with most functions, the two mixed partials are equal. (You should use this fact to check your answers.) Note the use of the chain rule for both first and second partial derivatives. Note that the product rule is needed for the second partial derivatives.

answers: $f_{xx} = -4(2x+3y)^{-2} - y^2\sin(xy)$, $f_{yy} = -9(2x+3y)^{-2} - x^2\sin(xy)$,

$f_{xy} = f_{yx} = -6(2x+3y)^{-2} - xy\sin(xy) + \cos(xy)$

Theorem 13.5. "Let *f* be a function of two variables, and assume that f_{xy} and f_{yx} are continuous at (x_0, y_0). Then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.

[For this section, you will not need to know about "Differentiation Under the Integral Sign".
You may find it interesting to read through the extended application "Analysis of a Rainbow".

Example F: For $g(x, y) = \ln(xy) + e^{xy}$ determine $\frac{\partial^2 f}{\partial y \partial x}$. (Just like regular derivatives, you need to know and be able to use the product rule, quotient rule and chain rule.) *answer*: $xye^{xy} + e^{xy}$

Example G: The surface area *S* of a human being, in m^2, is approximated by $S(W, H) = 0.202W^{0.425}H^{0.725}$, where W = weight in kg and H = height in m. Given my height of 1.83 m and ideal weight of 85 kg, describe how my surface area would be changing if I started to gain weight above my ideal.
In words, we need a rate of change for *S* with respect to *W* where *H* remains constant at 1.83. That is, we need a partial derivative. *answer*: $0.08585\left(85^{-0.575}\right)\left(1.83^{0.725}\right) \approx 0.0103$ m^2 per kg gained