Calculus 241, section 13.4 The Chain Rule

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Way back in Calculus I, we encountered the Chain Rule for functions of one variable: $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$.

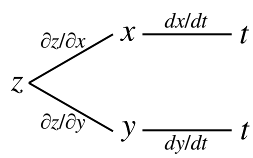
In section 13.3, we used this Chain Rule when focusing on one variable at a time:

$$(f \circ g)_x = (f_x \circ g) * g_x = \frac{\partial f}{\partial x}(g) * \frac{\partial g}{\partial x} \qquad\qquad (f \circ g)_y = (f_y \circ g) * g_y = \frac{\partial f}{\partial y}(g) * \frac{\partial g}{\partial y}.$$

Now we turn to compositions of functions of several variables, which get a little more complicated because we are juggling more than one variable.

a. Let $z = f(x, y)$, with parametric equations $x = g_1(t)$ and $y = g_2(t)$, i.e. $z = f(g_1(t), g_2(t))$.

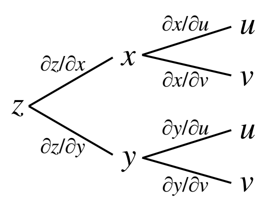
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} * \frac{dx}{dt} + \frac{\partial z}{\partial y} * \frac{dy}{dt}$$



b. Let $z = f(x, y)$, with parametric equations $x = g_1(u, v)$ and $y = g_2(u, v)$, i.e. $z = f(g_1(u, v), g_2(u, v))$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} * \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} * \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} * \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} * \frac{\partial y}{\partial v}$$



For those of you who have used tree diagrams in probability, the arithmetic is the same: multiply when moving from the root to the end of the branch, and add when moving down the branches on the right side.

For functions of three variables, we'll be drawing analogous diagrams and develop analogous formulas.

Example A. Given $z = \ln(3x - y^2)$, $x = \sqrt{t}$, $y = e^{t/2}$, find $\frac{dz}{dt}$.

Example B. Given $f(x, y) = 9 - x^2 - y^2$, $x = u\cos v$, $y = v\sin u$, find $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$.

Example C. Given $w = x^2 yz - ze^{xy}$, $x = 2t,\ y = \frac{1}{t},\ z = \sqrt{t}$ find $\frac{dw}{dt}$.

Example D. For $f(x,\ y,\ z) = x^2 + 3xy + z,\ x = u\cos v,\ y = v\sin u,\ z = uv$, find $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$.

Example E. An oil tanker has run aground and is leaking crude oil. Assuming (simplistically) calm waters, the oil slick will be circular. As the oil spreads outward from the tanker, the depth of the oil will decrease. Suppose the radius of the oil slick is increasing at a rate of 0.5 feet per minute and the depth of the oil on the water is decreasing at a rate of 0.5 inch per minute, what is the rate of change of the volume of the oil slick with respect to time?