Calculus 241, section 13.5 Directional Derivatives

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In section 13.3, we looked at partial derivatives.

$$\frac{\partial f(x, y)}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

With y treated as a constant, the tangent line goes in the direction of the x-axis. The text gives a more formal explanation, connecting $\frac{\partial f}{\partial x} = f_x$ to a tangent vector $\vec{i} + f_x(x_0, y_0)\vec{k}$.

Similar observations would apply to $\frac{\partial f}{\partial y} = f_y$.

But what if we wanted a tangent (derivative) in a direction other than one of the axes, for example in the direction of a unit vector $\mathbf{u} = \vec{u} = a_1 \vec{i} + a_2 \vec{j}$?

Definition 13.12. "Let f be a function on a set containing a disk D centered at (x_0, y_0), and let $\mathbf{u} = \vec{u} = a_1 \vec{i} + a_2 \vec{j}$ be a unit vector. Then the **directional derivative** of f at (x_0, y_0) in the direction of $\vec{u}$ …is defined by

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha_1, y_0 + ha_2) - f(x_0, y_0)}{h}$$

provided that this limit exists."

Note that if $\vec{u} = \vec{i} = 1\vec{i} + 0\vec{j}$, we get $D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = f_x(x_0, y_0)$,

and if $\vec{u} = \vec{j} = 0\vec{i} + 1\vec{j}$ we get $D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} = f_y(x_0, y_0)$.

It turns out that finding a directional derivative is relatively simple, involving the partial derivatives and the components of the directional vector.

Theorem 13.13. "Let f be differentiable at (x_0, y_0). Then f has a directional derivative at (x_0, y_0) in every direction. Moreover, if $\vec{u} = a_1 \vec{i} + a_2 \vec{j}$ is a unit vector, then

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)a_1 + f_y(x_0, y_0)a_2 \text{."}$$

The text has a four-line proof.

Example A. Given $f(x, y) = 9 - x^2 - y^2$ and $\vec{u} = \frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{j}$, find $D_{\vec{u}} f(1, 2)$. *answer*: $-\sqrt{3} - 2$

Note that the directional derivative is a formula that gives us a number value for slope at a particular point in a particular direction.

We can extend Theorem 13.13 to functions of three variables. If *f* is differentiable at (x_0, y_0, z_0), and $\vec{u} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ is a unit vector, then

$$D_{\vec{u}} f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)a_1 + f_y(x_0, y_0, z_0)a_2 + f_z(x_0, y_0, z_0)a_3.$$

Example B. Given $f(x, y, z) = \frac{x}{y}\ln(z)$ and $\vec{a} = \vec{i} - \sqrt{5}\,\vec{j} + \sqrt{3}\,\vec{k}$, find $D_{\vec{u}} f(2, 1, 3)$ in the direction of $\vec{a}$.

answer: $\frac{1}{3}\ln(3) + \frac{2\sqrt{5}}{3}\ln(3) + \frac{2\sqrt{3}}{9}$