Calculus 241, section 13.6-13.7 The Gradient, Tangent Plane Approximation

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Definition 13.14a. "Let *f* be a function of two variables that has partial derivatives at (x_0, y_0). Then the **gradient** of *f* at (x_0, y_0)…is defined by

$$\text{grad}\, f(x_0, y_0) = \nabla f(x_0, y_0) = f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j}\,."$$

Definition 13.14b. "Let *f* be a function of three variables that has partial derivatives at (x_0, y_0, z_0). Then the **gradient** of *f* at (x_0, y_0, z_0)…is defined by

$$\text{grad}\, f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)\vec{i} + f_y(x_0, y_0, z_0)\vec{j} + f_z(x_0, y_0, z_0)\vec{k}\,."$$

Example A. Given $f(x, y) = 9 - x^2 - y^2$, find a formula for the gradient of *f* and in particular find $\text{grad}\, f(1, 2)$.

Note that the gradient is a vector that gives us an indication of slope at a particular point on a surface.

Example B. Given $f(x, y, z) = \frac{x}{y}\ln(z)$ find a formula for the gradient of *f* and in particular find $\nabla f(2, 1, 3)$.

Theorem 13.15. "Let *f* be a function of two variables that is differentiable at (x_0, y_0).

a. For any unit vector **u** $[= \vec{u}]$, $D_{\vec{u}} f(x_0, y_0) = [\text{grad}\, f(x_0, y_0)] \bullet \vec{u}$.

b. The maximum value of $D_{\vec{u}} f(x_0, y_0)$ is $\|\text{grad}\, f(x_0, y_0)\|$.

c. If $\text{grad}\, f(x_0, y_0) \neq \vec{0}$, then $D_{\vec{u}} f(x_0, y_0)$, regarded as a function of $\vec{u}$, attains its maximum value when $\vec{u}$ points in the same direction as $\text{grad}\, f(x_0, y_0)$.

Another way of saying part c. is to picture the surface $z = f(x, y)$ as a hill. At the point (x_0, y_0, z_0) the steepest ***positive*** incline is in the direction of $\text{grad}\, f(x_0, y_0)$.

Example A extended. Given $f(x, y) = 9 - x^2 - y^2$, determine the direction in which *f* increases most rapidly at $(x, y) = (1, 2)$ and find the maximal directional derivative at $(x, y) = (1, 2)$.

Theorem 13.16 is the "function of three variables" version of Theorem 13.15.
"Let *f* be a function of three variables that is differentiable at (x_0, y_0, z_0).
a. For any unit vector **u** $[= \vec{u}]$ in space, $D_{\vec{u}} f(x_0, y_0, z_0) = [\text{grad } f(x_0, y_0, z_0)] \bullet \vec{u}$.
b. The maximum value of $D_{\vec{u}} f(x_0, y_0, z_0)$ is $\|\text{grad } f(x_0, y_0, z_0)\|$.
c. If $\text{grad } f(x_0, y_0, z_0) \neq \vec{0}$, then $D_{\vec{u}} f(x_0, y_0, z_0)$, regarded as a function of $\vec{u}$, attains its maximum value when $\vec{u}$ points in the same direction as $\text{grad } f(x_0, y_0, z_0)$.

We're not going to do an Example in class for Theorem 13.16. It would end up looking a lot like Example A extended, but with three variables instead of two.

Next topic: Do you remember level curves from section 13.1?
A level curve *C* was determined by setting $f(x, y)$ equal to a constant, i.e. $f(x, y) = c$. If we consider a point (x_0, y_0) on *C*, the directional derivative $D_{\vec{u}} f(x_0, y_0)$ in a direction tangent to *C* would equal 0, because the rate of change of *f* along the level curve is 0.
Using Theorem 13.15a we get, $D_{\vec{u}} f(x_0, y_0) = [\text{grad } f(x_0, y_0)] \bullet \vec{u} = 0$.
What do we know when the dot product of vectors is 0?

Theorem 13.17. "Let *C* be a level curve $f(x, y) = c$ of a function *f*. Let (x_0, y_0) be a point on *C*, and assume that *f* is differentiable at (x_0, y_0). If *C* is smooth and $\text{grad } f(x_0, y_0) \neq \vec{0}$, then $\text{grad } f(x_0, y_0)$ is normal [perpendicular] to *C* at (x_0, y_0)."

Example C. Assuming that the curve $x^2 + 3xy + y - 9 = 0$ is smooth, find a unit vector that is perpendicular to the curve at the point (1, 2).

Definition 13.18. "Let *f* be differentiable at a point (x_0, y_0, z_0) on a level surface *S* of *f*.
If $\operatorname{grad} f(x_0, y_0, z_0) \neq \vec{0}$, then the plane through (x_0, y_0, z_0) whose normal is $\operatorname{grad} f(x_0, y_0, z_0)$ is the plane **tangent** to *S* at (x_0, y_0, z_0). Any vector that is perpendicular to this tangent plane is said to be **normal** to *S*."

From Definition 13.18, we get $\operatorname{grad} f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)\vec{i} + f_y(x_0, y_0, z_0)\vec{j} + f_z(x_0, y_0, z_0)\vec{k}$ is normal to the tangent plane. We can easily find the equation of the tangent plane using this normal and the point (x_0, y_0, z_0) which is on the tangent plane.

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

Example B extended. Find an equation of the plane tangent to the surface $\frac{x}{y}\ln(z) = 3$ at the point $(3, 1, e)$.

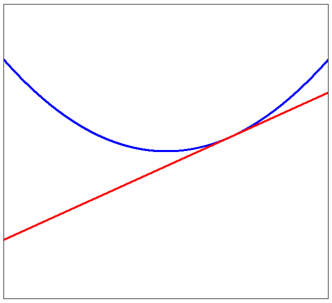
Given a function of two variables $f(x, y)$ which is differentiable at a point (x_0, y_0), the process for finding an equation for a tangent plane is only slightly more complicated. We have to think of the graph of *f* as a level surface for the function $g(x, y, z) = f(x, y) - z$. Since $\operatorname{grad} g(x_0, y_0, z_0) = f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j} - 1\vec{k}$ is normal to the tangent plane, we can use this normal and the point $(x_0, y_0, f(x_0, y_0))$ which is on the tangent plane.

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0$$

Example C extended. Given $f(x, y) = x^2 + 3xy + y$ find an equation of the plane tangent to the graph of *f* at the point (1, 2, 9).

Here begins section 13.7

First a quick digression back to Calc I.



We were able to use the line tangent to a graph at a point (x_0, y_0) to estimate function values for $f(x_0+h)$. Essentially,
the point on the curve $(x_0+h, f(x_0+h))$ was approximated by
a point on the tangent line $(x_0+h, f'(x_0)*x_0+b)$.

As long as the value of h was pretty close to 0, the estimate was pretty good. The further away we travel from x_0, the less reliable the estimate.

But, we didn't have to know the equation of the tangent line to come up with our estimate! (That is, we didn't need to find the value of b in the tangent line equation.) Instead,

$$f'(x_0) \approx \frac{f(x_0+h)-f(x_0)}{h} \quad \Rightarrow \quad f(x_0+h) \approx f(x_0)+f'(x_0)*h.$$

The text extends this concept to multivariable functions. (Read the formal proof and explanation for yourself.) We can use the tangent plane to estimate function values close to a point by knowing only the coordinates of the initial point, and the values of the partial derivatives at that point. (That is, we won't need to find the equation of the tangent plane.)

$$f(x_0+h, y_0+k) \approx f(x_0, y_0)+f_x(x_0, y_0)*h+f_y(x_0, y_0)*k$$

$$f(x_0+h, y_0+k, z_0+l) \approx f(x_0, y_0, z_0)+f_x(x_0, y_0, z_0)*h+f_y(x_0, y_0, z_0)*k+f_z(x_0, y_0, z_0)*l$$

Example B once again. Given $f(x, y, z)=\frac{x}{y}\ln(z)$, use tangent plane approximation to estimate the value of $f(3.1, 0.9, e+0.1)$. *answer*: ≈ 3.51

Why we would we want an estimate?
Why not just use the equation and a calculator?

In the real world, there is often not an equation, but just data that describe a situation, and an approximation is the best that can be done.

For example, the Federal Reserve Board has to use data about current economic conditions, along with observations about how various factors are changing to predict future conditions, and make policy decisions.