Calculus 241, section 13.8 Extreme Values of Multivariable Functions

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Just like we had a first-derivative test and a second-derivative test to identify maxima and minima of functions of one variable, we'll use versions of first partial and second partial derivatives to determine maxima and minima of functions of more than one variable.

Definition 13.19 (non-technical version). Given a function of two variables *f* whose graph is a surface, and a region *R* in the domain of *f*, the function has a maximum [minimum] value if there is a highest [lowest] point on the surface. The function has a relative maximum [relative minimum] value if there is a highest [lowest] point within an open disk *D* (local neighborhood) within the domain of *f*.

The first derivative test for functions of more than one variable looks very much like the first derivative test we have already used for functions of one variable.

Theorem 13.20. If $f(x, y)$ has a relative maximum or minimum at values (x_0, y_0) then all partial derivatives will equal 0 at that point. That is,

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = 0 \qquad \frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, y_0) = 0$$

or equivalently $\text{grad } f(x_0, y_0) = \nabla f(x_0, y_0) = f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j} = \vec{0}$.

Example A: Find the possible values of x and y at which $f(x, y) = x^2 + 2y^3 + 4x - 6y + 9$ has a relative maximum or minimum. That is, find all critical points. *answers*: (–2, –1); (–2, 1)

Example B: Find the values of x, y and z at which $f(x, y) = x^2 + 4xy + y^2 - 12y$ might have a relative maximum or minimum. *answer*: (4, –2, 12)

Notice that the examples above asked for *possible* values. The first derivative test by itself is inconclusive. The second derivative test for functions of more than one variable is a good bit more complicated than the one used for functions of one variable. We'll apply it only to functions of two variables.

Theorem 13.21. Assume *f* has continuous second derivatives at in a disk centered at a critical point (x_0, y_0).

First calculate $D = \frac{\partial^2 f}{\partial x^2} * \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\, \partial y}\right) = f_{xx} * f_{yy} - (f_{xy})^2$.

One way to remember this: It is the determinant of the 2×2 matrix $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$.

Then, given a point (x_0, y_0) which represents a possible extreme:

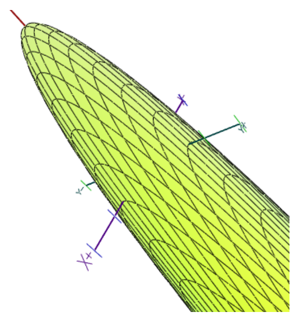
a. If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$ or $f_{xyy}(x_0, y_0) < 0$, then *f* has a relative maximum at (x_0, y_0).

b. If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$ or $f_{xyy}(x_0, y_0) > 0$, then *f* has a relative minimum at (x_0, y_0).

c. If $D(x_0, y_0) < 0$ then *f* has a saddle point at (x_0, y_0).

d. If $D(x_0, y_0) = 0$, then the test is inconclusive—we don't know what's happening at (x_0, y_0).

Example C (Example G from 13.1 revisited): The 3-D graph of the function $f(x, y) = 9 - x^2 - y^2$ shows a relative maximum. We'll use the tests above to determine the location and verify that it is a maximum.



Example D: Use first and second derivative tests to determine points where $f(x, y) = 3x^3 + y^2 - 9x - 6y + 1$ has relative extrema. *answers*: (–1, 3, –2) saddle point; (1, 3, –14) relative minimum

Example E: Use first and second derivative tests to determine points where $f(x, y) = xy - x^2 - y^2 - x - 4y + 4$ has relative extrema. *answer*: (–1,–2, 11) relative maximum

Theorem 13.22. "Let *R* be a closed, bounded set in a plane, and let *f* be continuous on *R*, Then *f* has both a maximum value and a minimum value on *R*."

Theorem 13.22 is analogous to the Maximum-Minimum Theorem for functions of one variable covered in section 4.2.

Example D revisited: Find the extreme values of $f(x, y) = 3x^3 + y^2 - 9x - 6y + 1$ on the region *R* defined by $0 \le x \le 1,\ 0 \le y \le 3$. *answers*: maximum value is 1; minimum value is –14