Calculus 241, section 13.9 Lagrange Multipliers

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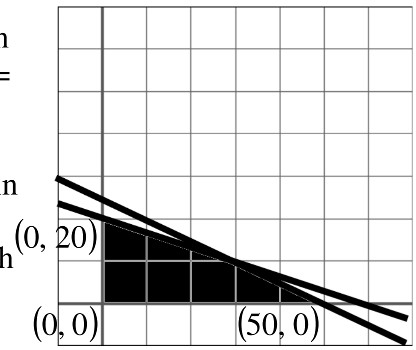
In a finite mathematics course (Math 110 here at UMCP) students encounter linear programming. A typical word problem looks like this:

> A company manufactures two types of desks. Let x = the number of steel desks and let y = the number of wood desks. The profit for steel desks is \$80 each, and the profit for wood desks is \$175. The company wants to maximize its profit. Each steel desk requires 2 hours of assembly and 1 hour of finishing. Wood desks require 4 hours of assembly and 3 hours of finishing. The company has 100 work-hours available for assembly and 60 work-hours available for finishing.



The company's goal (i.e. objective) is to maximize profit: the objective function is $P = 80x + 175y$. In theory, this function has no maximum: make more desks = make more money. In the real world, however, there are limitations (i.e. constraints): the number of employees and therefore the number of desks that can be made has an upper limit. The hours available for assembly is expressed in the assembly constraint $2x + 4y \le 100$. The hours available for finishing is expressed in the finishing constraint $x + 3y \le 60$. In addition, the number of each type of desk made cannot be negative: $x \ge 0$ and $y \ge 0$. The "system of constraints" which illustrates the "feasible set" is graphed to the right, with corners labeled.

If we were to graph the *level curves* for the profit function $P = 80x + 175y$, they would appear as a series of parallel lines: all with the same slope but representing varying levels of profit. The maximum possible (i.e. feasible) profit is represented by the level curve where $P = 4150$ that passes through the corner (30, 10).

You won't be asked to do a linear programming question in this class, but you will need some of the same algebra skills, such as solving a system of equations.

Rather, in calculus, while we're still looking for some optimum value (maximum or minimum), neither the objective nor the constraints are likely to be linear functions, and we'll need somewhat more involved methods of finding the maximum or minimum of the objective, within the given constraints.

Specifically, if we are looking for an intersection, at a point, (x_0, y_0), of an extreme of a function $f(x, y)$ with a level curve $g(x, y) = c$, then from Theorem 13.17 we have that the normals to the two curves are parallel. That is $\text{grad}\, f(x_0, y_0)$ is a multiple of $\text{grad}\, g(x_0, y_0)$. (For a more formal and more technically complete explanation, see your text at the beginning of section 13.9 and the proof of Theorem 13.23.)

Theorem 13.23. "Let f and g be differentiable at (x_0, y_0). Let C be the level curve $g(x, y) = c$ that contains (x_0, y_0). Assume C is smooth and that (x_0, y_0) is not an endpoint of the curve. If $\text{grad}\, g(x, y) \ne \vec{0}$ and if f has an extreme value on C at (x_0, y_0), then there is a number λ such that

$$\text{grad}\, f(x, y) = \lambda\, \text{grad}\, g(x, y)."$$

1) The number λ is called a **Lagrange multiplier**.
2) The process we'll develop below will work for functions of three variables as well as for functions of two variables, although the algebra will usually become more complicated.

Given an objective function f and a constraint function g the process looks like this:

Identify the objective function f — it's the one that needs to be maximized or minimized — and the constraint.

Assume that f has an extreme value on the level curve $g(x, y) = c$.

Set up $\text{grad}\, f(x, y) = \lambda\, \text{grad}\, g(x, y)$ and solve the resulting system of equations for one variable in terms of the other(s).

(It will often be best to first solve for λ and set these formulas equal to each other, using a series of substitutions to find the rest of the values.)

Substitute into the constraint $g(x, y) = c$.

If the constraint involves a region *R*, look for critical values in the interior of the region.

Calculate the value of *f* at each point (x, y) that arises from the above to identify the maximum and minimum.

Example A: Find the maximum and minimum values of $f(x, y) = 2x^2 + y^2 + 3$ such that $x + y = 9$.
answer: minimum $f(3, 6) = 57$, no maximum

Conceptual explanation: The graph of $f(x, y) = 2x^2 + y^2 + 3$ is a 3-D surface, for which there are an infinite number of *z*-values. (It's a parabolic-elliptical shape with a minimum at (0, 0, 3)).
The graph of $x + y = 9$ is a plane parallel to the *z*-axis, for which there are an infinite number of *z*-values.
The intersection of these two contains an infinite number of points. (It's a parabola parallel to the *z*-axis.)
The smallest *z*-value for any of these intersections is found at the point (3, 6, 57).
Like any parabola opening up, there is no maximum.

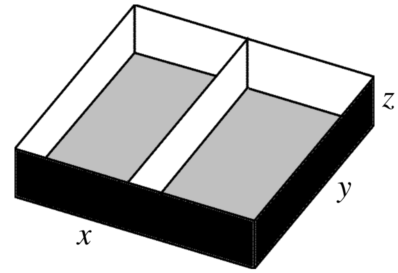
Example B: Find the extreme values of $f(x, y) = x^3 + 3y^2$ on the disk $x^2 + y^2 \le 9$.
answer: minimum $f(0, -3) = -27$, maximum $f(-3, 0) = f(3, 0) = 18$

By Theorem 13.22 (Maximum-Minimum Theorem) we know that *f* will have both a minimum and a maximum on the disk, either on a boundary or in the interior.

[Recall Example D revisited from section 13.8: Find the extreme values of $f(x, y) = 3x^3 + y^2 - 9x - 6y + 1$ on the region *R* defined by $0 \le x \le 1$, $0 \le y \le 3$. We'll follow much the same process here, checking both interior (with the first derivative test) and boundary points (with a Lagrange multiplier).]

Conceptual explanation: The graph of $f(x, y) = x^3 + 3y^2$ is a 3-D surface, for which there are an infinite number of *z*-values. (It has an orchid-flower shape with neither relative minimum nor maximum.)
The graph of $x^2 + y^2 \le 9$ is a solid cylinder parallel to the *z*-axis, for which there are an infinite number of *z*-values.
The intersection of these two contains an infinite number of points.
The smallest *z*-value for any of these intersections is found at the point (0, –3, –27).
The largest *z*-value for any of these intersections is found at the points (–3, 0, 18) and (3, 0, 18).

Example C: We want to make a rectangular open box with one partition in the middle, as illustrated in the picture, from 162 in^2 of cardboard. Find the dimensions that would maximize the volume. (Okay, this isn't much like a real-world problem. If you'd rather, you can change it to "cargo container and sheet metal".)



Answer: 9 in by 6 in by 3 in

In this Example, the Lagrange multiplier has a physical interpretation: λ = marginal volume with respect to surface area = $\lambda = \frac{-xz}{x+3z} = \frac{-9*3}{9+3*3} = -\frac{27}{18} = -\frac{3}{2}$. At the optimal level, changes in the dimensions, ***within the given constraint***, would decrease the volume.