Math 241 Chapter 13 **Dr. Justin O. Wyss-Gallifent**

§13.1 Functions of Several Variables

1. Definition: A function like $f(x,y)$, $f(x,y,z)$, $g(s,t)$ etc.
2. Definition of the graph of a function of two variables and classic examples like: Plane, paraboloid, cone, parabolic sheet, hemisphere.
3. Definition of level curve for $f(x,y)$ and level surface for $f(x,y,z)$.
4. Graphs of surfaces which are not necessarily functions: Sphere, ellipsoid, cylinder sideways parabolic sheet like $y=x^2$, double-cone.

§13.2 Limits and Continuity

1. Nothing much said other than $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ asks what $f(x,y)$ approaches as (x,y) gets closer to (x_0,y_0).

§13.3 Partial Derivatives

1. Defn: We can define *the partial derivative of* $f(x,y)$ *with respect to* x, denoted $\frac{\partial f}{\partial x}$ or f_x, as the derivative of f treating all variables other than x as constant. Similarly for any variable for any function.
2. For $f(x,y)$ it turns out f_x and f_y give the slopes of the lines tangent to the graph of $f(x,y)$ at the point (x,y) in the positive x and positive y directions respectively. A picture can clarify.
3. Higher derivatives will also be used but there are some points to note:
   (a) f_{xy} means $(f_x)_y$ so first take the derivative with respect to x and then y.
   (b) $\frac{\partial^2 f}{\partial x \partial y}$ means $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ so first take the derivative with respect to y and then x.
   (c) $\frac{\partial^2 f}{\partial x^2}$ means x both times.
   (d) It turns out that 99% of the time the order doesn't matter so for example $f_{xy} = f_{yx}$.

§13.4 The Chain Rule

1. Consider: For example if f is a function of x and y which are both functions of s and t then really f is a function of s and t and so $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ make sense. How to find them?
2. The chain rule says:
   (a) First draw the tree diagram.
   (b) For each route from the starting to ending variable write down the product of the derivatives along that path.
   (c) Add the paths.
3. The chain rule is good for related rates problems when multiple rates are given and one rate is needed.

§13.5 Directional Derivative

1. Intro: We saw that f_x and f_y (for example) give derivatives in specific directions (the $\hat{\imath}$ and $\hat{\jmath}$ directions) and so what if we asked for the derivative (slope) in another direction?

2. Defn: If $\bar{u} = a\,\hat{\imath} + b\,\hat{\jmath}$ is a unit vector then *the directional derivative of f in the direction of $\bar{u}$* is $D_{\bar{u}}f = af_x + bf_y$. If we have 3D then $+cf_z$ on the end. Sometimes we use the term "directional derivative" when the direction is not a unit vector so we must make it a unit vector first.

3. A good analogy is that $f(x, y, z)$ is temperature and $D_{\bar{u}}f$ gives us temperature change (slope) in a specific direction.

§13.6 The Gradient

1. Defn: The gradient of f, denoted grad f or ∇f, is defined as $\nabla f = f_x\,\hat{\imath} + f_y\,\hat{\jmath}$ and $+f_z\,\hat{k}$ in 3D.

2. Properties:

   (a) For any $\bar{u}$ we see $D_{\bar{u}}f = \bar{u} \cdot \nabla f$.

   (b) Since $D_{\bar{u}}f = \bar{u} \cdot \nabla f = ||\bar{u}||||\nabla f|| \cos\theta = ||\nabla f|| \cos\theta$ we see that the directional derivative is maximum when $\theta = 0$ which shows that the gradient points in the direction of maximum directional derivative.

   (c) It also shows that the actual value of the maximum directional derivative is $||\nabla f||$.

   (d) In the $2D$ case ∇f is perpendicular to the level curve for $f(x, y)$ at (x, y). If we want a vector perpendicular to the graph of a function $f(x)$ we need to rewrite as $y = f(x)$ then $f(x) - y = 0$ and then the graph of the function is the level curve for $g(x, y) = f(x) - y$ and we use ∇g.

   (e) In the $3D$ case ∇f is perpendicular to the level surface for $f(x, y, z)$ at (x, y, z). If we want a vector perpendicular to the graph of a function $g(x, y)$ we need to rewrite as $z = g(x, y)$ then $g(x, y) - z = 0$ and then the graph of the function is the level surface for $f(x, y, z) = g(x, y) - z$ and we use ∇f.

3. Tangent Plane: From (e) above we see that the plane which is tangent to the level surface for $f(x, y, z)$ at (x_0, y_0, z_0) has equation $f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$.

§13.7 Tangent Plane Approximations and Differentials

1. Tangent Plane Continued from 13.6: Moreover if this level surface is the graph of a function $g(x, y)$ then this becomes $g_x(x - x_0) + g_y(y - y_0) - 1(z - g(x_0, y_0)) = 0$. If we use f instead of g we can solve for z to get $z = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$.

2. Idea: If we have the graph of a function $f(x, y)$ and take the tangent plane at a specific point $(x_0, y_0, f(x_0, y_0))$ then the tangent plane will be close to the function if we stay near (x_0, y_0).

3. Formula: $f(x_0 + h, y_0 + k) \approx f(x_0, y_0) + f_x(x_0, y_0)h + f_y(x_0, y_0)k$

4. Generalization: If we have $f(x, y, z)$ instead then there really isn't a tangent plane but more of a tangent space. This is harder to visualize but the formula is fairly obvious:
   $f(x_0 + h, y_0 + k, z_0 + l) \approx f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)h + f_y(x_0, y_0, z_0)k + f_z(x_0, y_0, z_0)l$

§13.8 Extreme Values

1. Defn: Relative maximum/minimum/extremum for $f(x,y)$. Method:

   (a) First find where both f_x and f_y are zero or one is undefined. Those are the *critical points*.

   (b) Find the discriminant $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2$ and then for each critical point:

   - If $D(x,y) < 0$ then (x,y) is a saddle point.
   - If $D(x,y) > 0$ and $f_{xx}(x,y) < 0$ then (x,y) is a relative maximum.
   - If $D(x,y) > 0$ and $f_{xx}(x,y) > 0$ then (x,y) is a relative minimum.

   Good examples: $f(x,y) = x^2 + 2y^2 - 6x + 8y + 1$ and $f(x,y) = 3x^2 - 3xy^2 + y^3 + 3y^2$.

2. Defn: Absolute m/m/e of $f(x,y)$ on a closed and bounded region R. Method:

   (a) Find all CP for $f(x,y)$ which are inside the region. Take f of those.

   (b) Find the maximum and minimum of f on the edge of the region. Usually this involves combining f with the equation for the region (sometimes part by part) and then getting f in a form where we can see what the max and min would be.

   (c) Pick out the largest and smallest values from the prevous two steps.

   Good examples: $f(x,y) = x^2 - y^2$ with $\frac{x^2}{4} + y^2 \leq 1$ and $f(x,y) = 3x - y$ on the triangle with vertices $(0,0)$, $(0,3)$ and $(6,0)$.

§13.9 Lagrange Multipliers

1. Idea: If (x,y) are constrained by a level curve $g(x,y) = c$ and we want to find the maximum of $f(x,y)$ how do we do it?

2. Thm: If a max/min occurs at (x,y) then $\nabla f = \lambda \nabla g$ at that point so the method is:

   (a) We set those equal and solve those along with the constraint. In other words we solve the system: $f_x = \lambda g_x$, $f_y = \lambda g_y$ and $g(x,y) = c$.

   (b) The result are potential winners. We take each (x,y) we get and plug it into f, picking out the largest and smallest.

   Good Examples: $f(x,y) = 2x + 3y$ with $x^2 + y^2 = 9$, $f(x,y) = xy$ with $(x-1)^2 + y^2 = 1$ and $f(x,y) = x^2 + y^2$ with $2x + 6y = 10$.