Calculus 241, section 14.1 Double Integrals

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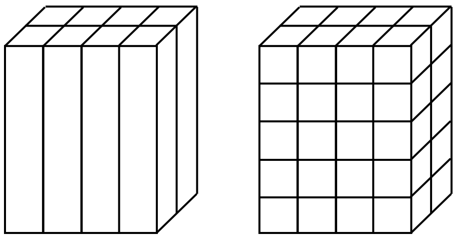
For this section, I'll leave much of the technical stuff (explanations, Definitions 14.1 and 14.2) for your own reading the text. In summary: Take the "rectangles under a curve" idea of a Riemann sum in two dimensions, and expand it into three dimensions. "Area under a curve" is expanded into "volume under a surface".



The underlying theory involves using a series of parallelpipeds (essentially a three-dimensional rectangle, or box), drawn extending up from a two-dimensional area *A*, as the basis for a double integral: $\lim_{\Delta x \to 0}, \lim_{\Delta y \to 0}$.

We'll spend most of this Lecture concentrating on the part most important to you: evaluating double integrals.

Example A: Evaluate $\int x^2 + 4xy + y^2 - 12y + 2 \, dx$.

The principle here is the same as for finding a partial derivative with respect to *x*: all other variables are treated like constants. *answer*: $\frac{1}{3}x^3 + 2x^2y + xy^2 - 12xy + 2x + C$

Do-it-yourself example: $\int x^2 + 4xy + y^2 - 12y + 2 \, dy = x^2y + 2xy^2 + \frac{1}{3}y^3 - 6y^2 + 2y + C$

Example A extended: Evaluate $\int_0^1 \int_1^2 x^2 + 4xy + y^2 - 12y + 2 \, dx \, dy$.

The principle here is the same as for finding a mixed partial (second) derivative with respect to *x* then with respect to *y*. *answer*: $\frac{5}{3}$

Do-it-yourself version: $\int_1^2 \int_0^1 x^2 + 4xy + y^2 - 12y + 2 \, dy \, dx$ yields the same result.

Example B: Find the double integral $\iint_R x\sqrt{2x^2+3y}\ dy\ dx$ over the rectangular region $0 \le x \le 1,\ 1 \le y \le 2$.

This is really the same type of integral as in Example A extended. Be careful that you put the boundaries of integration into the correct integral. *answer*: $\frac{1}{45}\left(8^{5/2} - 5^{5/2} - 6^{5/2} + 3^{5/2}\right) \approx 1.1672$

Two notes on Example B:

1) Integration by parts may show up in text exercises, but since it's an even more labor-intensive process than substitution, we won't have time to do an example in lecture. (However, see the Bonus example below.)

2) We could have switched this around to do the more involved integration by substitution (*dx*) first, as long as we kept the boundaries of integration straight, because the region *R* is a rectangle which is both vertically and horizontally simple.

$$\int_0^1 \int_1^2 x\left(2x^2+3y\right)^{1/2}\ dy\ dx = \int_1^2 \int_0^1 x\left(2x^2+3y\right)^{1/2}\ dx\ dy$$

However, it turns out to be a good bit harder if we do the switch. Rule of thumb: Do the easier integration first.

What do we mean by "vertically simple" and "horizontally simple"?

Definition 14.3: "**a.** A plane region *R* is **vertically simple** if there are two continuous functions g_1 and g_2 on an interval [*a*, *b*] such that $g_1(x) \le g_2(x)$ for $a \le x \le b$ and such that *R* is the region between the graphs of g_1 and g_2 on [*a*, *b*]." Part **b.** provides almost exactly the same definition for **horizontally simple**, with $c \le y \le d$.
"**c.** A plane region *R* is **simple** if it is both vertically simple and horizontally simple."

Also, see Theorem 14.4 in the text.

Example C: Find the volume under the surface $z = \dfrac{1}{2xy}$ and above the rectangle $1 \le x \le e$, $1 \le y \le e^2$. *answer*: 1

This is really the same type of integral as in Example A extended and Example B. In other words, your exam could ask the same question in any one of these three ways.

Example D: Evaluate $\int_0^1 \int_{-x}^{x^2} x^2 + 3xy + 2y \, dy \, dx$. *answer*: $\frac{23}{120}$

Example E: Evaluate $\iint_R x+2y\;dy\;dx$, where R is the closed region between $y=2x$ and $y=x^2$. *answer*: $\frac{28}{5}$

Example F: Evaluate $\int_0^{16}\int_{\sqrt{y}}^{4}\sqrt{x^3+4}\;dx\;dy$. *answer*: $\frac{2}{9}\left(68^{3/2}-4^{3/2}\right)$

Note that, as in the Examples above, we can switch the order of integration around only because the region R is both vertically and horizontally simple.

Bonus Example: Find the volume under the surface $z = \frac{1}{x+y}$ and above the rectangle $1 \le x \le e$, $1 \le y \le e$.

Set up the necessary double integral: $\int_1^e \int_1^e \frac{1}{x+y}\, dx\, dy$.

Integrate with respect to x using substitution: $u = x + y, \quad du = dx$.

$$\int_1^e \int \frac{1}{u}\, du\, dy = \int_1^e \ln u\, dy \quad \Rightarrow \quad \int_1^e \int_1^e \frac{1}{x+y}\, dx\, dy = \int_1^e \left[\ln(x+y)\right]_1^e dy = \int_1^e \left[\ln(e+y) - \ln(1+y)\right] dy$$

(We won't ask you to do something like the rest of this example on an exam – it would take you too much time because it requires a clever use of integration by parts, as well as "partial fractions".)

The next few lines apply integration by parts to a generic model, using c, which is then used below where "c" is replaced by e and 1 respectively.

Rewrite $\int \ln(c+y)\, dy = \int \ln(c+y) * 1\, dy$,

then use integration by parts: $dv = 1\, dy$, $v = y$, $u = \ln(c+y)$, $du = \frac{1}{c+y}\, dy$.

So $\int \ln(c+y)\, dy = \ln(c+y) * y - \int \frac{y}{c+y}\, dy$

(Here's where we use the fact that $\frac{y}{c+y} = 1 - \frac{c}{c+y}$. This is the "partial fractions" part.)

$$\int \ln(c+y)\, dy = y\ln(c+y) - \int 1 - \frac{c}{c+y}\, dy$$
$$= y\ln(c+y) - y + c\ln(c+y)$$

Next we apply this result to our original integration.

$$\int_1^e \ln(e+y) - \ln(1+y)\, dy = \left[y\ln(e+y) - y + e\ln(e+y)\right]_1^e - \left[y\ln(1+y) - y + 1\ln(1+y)\right]_1^e$$

From here on out it's a lot of algebraic evaluation.

$$\int_1^e \ln(e+y) - \ln(1+y)\, dy$$
$$= \left[e\ln(e+e) - e + e\ln(e+e)\right] - \left[1\ln(e+1) - 1 + e\ln(e+1)\right] - \left[e\ln(1+e) - e + 1\ln(1+e)\right] + \left[1\ln(1+1) - 1 + 1\ln(1+1)\right]$$
$$= \left[e\ln(2e) - e + e\ln(2e)\right] - \left[\ln(e+1) - 1 + e\ln(e+1)\right] - \left[e\ln(1+e) - e + \ln(1+e)\right] + \left[\ln(2) - 1 + \ln(2)\right]$$

distributing the subtraction:

$$= e\ln(2e) - e + e\ln(2e) - \ln(e+1) + 1 - e\ln(e+1) - e\ln(1+e) + e - \ln(1+e) + \ln(2) - 1 + \ln(2)$$

using logarithm properties:

$$= e\ln(2) + e\ln(e) - e + e\ln(2) + e\ln(e) - \ln(e+1) + 1 - e\ln(e+1) - e\ln(1+e) + e - \ln(1+e) + \ln(2) - 1 + \ln(2)$$
$$= e\ln(2) + e - e + e\ln(2) + e - \ln(e+1) + 1 - e\ln(e+1) - e\ln(1+e) + e - \ln(1+e) + \ln(2) - 1 + \ln(2)$$

using commutativity of addition then combining like terms:

$$= e - e + e + e + e\ln(2) + e\ln(2) - e\ln(e+1) - e\ln(1+e) - \ln(e+1) - \ln(1+e) + \ln(2) + \ln(2) + 1 - 1$$
$$= 2e + 2e\ln(2) - 2e\ln(e+1) - 2\ln(e+1) + 2\ln(2)$$