Calculus 241, section 14.2 Double Integrals in Polar Coordinates

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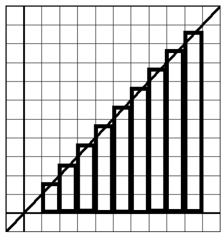
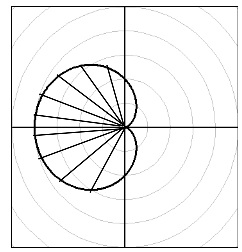
We need a little bit of theory and definition before we get to examples.

In chapter 5, we took rectangles and created a Riemann sum. In chapter 10, we took pie-shaped sectors and followed a similar process to find areas between curves in polar coordinates.





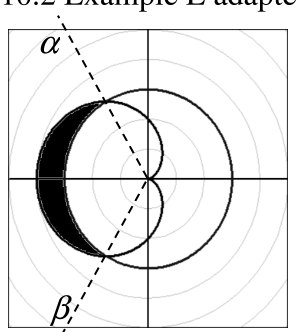
In section 14.1, the theory involved using a series of parallelpipeds (essentially a three-dimensional rectangle) as the basis for a double integral. Now, when we consider polar coordinates, we're using pie-shaped sectors extended up into the third dimension.

Combining this concept with the idea that (x, y) coordinates can be expressed in terms of polar values as $(r\cos\theta,\ r\sin\theta)$, we get Theorem 14.5.

"Suppose that h_1 and h_2 are continuous on $[\alpha, \beta]$, where $0 \le \beta - \alpha \le 2\pi$, and that $0 \le h_1(\theta) \le h_2(\theta)$ for $\alpha \le \theta \le \beta$. Let R be the region between the polar graphs of $r = h_1(\theta)$ and $r = h_2(\theta)$ for $\alpha \le \theta \le \beta$. If f is continuous on R, then $\iint_R f(x, y)\, dA = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\ r\sin\theta) * r\, dr\, d\theta$."

When determining the limits of integration for a polar double integral, we have to think in the same terms as we did when finding "area between polar curves".

10.2 Example E adapted: Suppose that our region R is the area that lies inside the cardioid $r = 2 - 2\cos\theta$ and outside the circle $r = 3$.



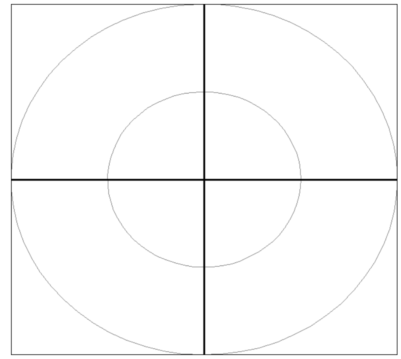
For the first integration (dr), θ is held constant. In designating the r boundaries of integration, we must think *radially*. Although the circle goes "above" the cardioid, we need to determine "outside" versus "inside". Thus, we must set the lower limit of integration to $r = h_1(\theta) = 3$ and the upper limit of integration to $r = h_2(\theta) = 2 - 2\cos\theta$.

To find the boundaries for the second integration ($d\theta$), find the value of θ for the two points of intersection:

$$2 - 2\cos\theta = 3 \quad \Rightarrow \quad -2\cos\theta = 1 \quad \Rightarrow \quad \cos\theta = -\frac{1}{2} \quad \Rightarrow \quad \theta = \frac{2\pi}{3} \text{ and } \frac{4\pi}{3}.$$

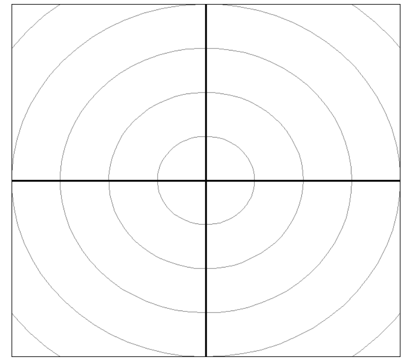
So the limits for the second integration are $\alpha = \frac{2\pi}{3}$ and $\beta = \frac{4\pi}{3}$.

Example A: Given $\iint\limits_R x\, dA$, where R is the region bounded by the circle $r = \sin\,\theta$, first express as an iterated integral in polar coordinates, then evaluate. *answer*: 0

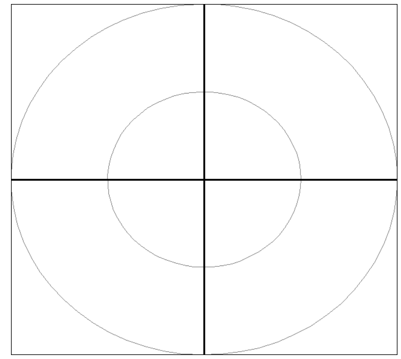


Our answer makes sense given the picture: "positive" and "negative" volume. (A two-dimensional analogy would be $\int_0^{2\pi} \sin\theta\, d\theta = 0$.) In Example A, we're integrating $f(x, y) = x$ [horizontal] over the region $r = \sin\,\theta$ [horizontally symmetric].

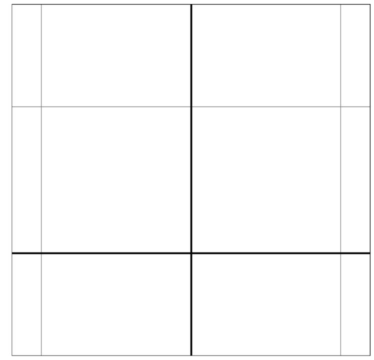
Example B: Find the volume V of the solid region bounded by the planes $z = 0$, $z = 6$, and the cylinder $r = 3 \sin \theta$. *answer*: $\frac{27\pi}{2}$



Example C: Find the volume V of the solid region bounded above by the planes $z = y$, on the sides by the cylinder $x^2 + y^2 = y$ and below by the xy-plane. *answer*: $\frac{\pi}{8}$

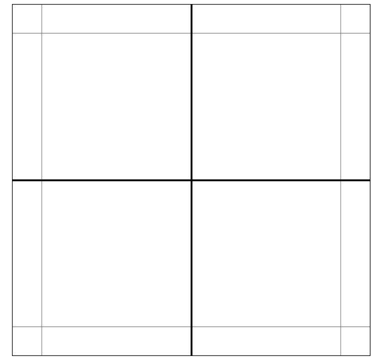


Example D: Given $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \left(x^2+y^2\right)^{3/2}\, dy\, dx$, change to an iterated integral in polar coordinates then evaluate. *answer*: $\frac{\pi}{5}$



We've run across a very convenient fact a couple of times so far: x^2+y^2 converted to polar coordinates simplifies very nicely: $x^2+y^2=[r\cos\theta]^2+[r\sin\theta]^2=r^2$.

Example E: Find the volume V of the solid region bounded above by the surface $z = e^{x^2+y^2}$, on the sides by the cylinder circle $x^2 + y^2 = 1$, and below by the xy-plane. *answer*: $\pi(e - 1)$



Note that we would not have been able to use symmetry if it were not for the fact that both the upper surface and the lower surface are symmetric with respect to both the x-axis and the y-axis.