Calculus 241, section 14.4 Triple Integrals

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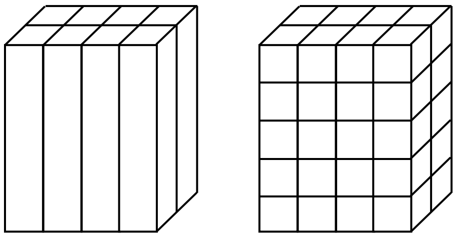
We'll start with a little bit of theory before we get to Examples.

For section 14.1, the theory involved using a series of parallelpipeds (essentially a three-dimensional rectangle, or box), drawn extending up from a two-dimensional area A, as the basis for a double integral: $\lim\limits_{\Delta x \to 0}, \lim\limits_{\Delta y \to 0}$.



Now, in section 14.4, we'll consider about a series of parallelpipeds, filling up a three-dimensional volume V (think "cube"), as the basis for a triple integral: $\lim\limits_{\Delta x \to 0}, \lim\limits_{\Delta y \to 0}, \lim\limits_{\Delta z \to 0}$.

As we did for section 14.1, we'll leave much of the technical stuff (Theorems 14.7, Definition 14.8, and explanations) to you to read in the text. A quick look at Theorem 14.9 and its implications will be helpful in getting a visual view of how we'll actually evaluate triple integrals.

"Let D be the solid region between the graphs of two continuous functions F_1 and F_2 on a vertically or horizontally simple region R in the xy-plane, and let f be continuous on D. Then

$$\iiint_D f(x, y, z)\, dV = \iint_R \left(\int_{F_1(x,y)}^{F_2(x,y)} f(x, y, z)\, dz \right) dA."$$

"If R is the vertically simple region between the graphs of g_1 and g_2 on $[a, b]$, we evaluate …

$$\iiint_D f(x, y, z)\, dV = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} \left(\int_{F_1(x,y)}^{F_2(x,y)} f(x, y, z)\, dz \right) dy \right] dx."$$

"Similarly, if R is the horizontally simple region between the graphs of h_1 and h_2 on $[c, d]$, … we obtain

$$\iiint_D f(x, y, z)\, dV = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} \left(\int_{F_1(x,y)}^{F_2(x,y)} f(x, y, z)\, dz \right) dx \right] dy."$$

Example A: Evaluate $\int_{-1}^{2}\int_{1}^{3}\int_{0}^{4} x^2 yz^3 \; dz\; dy\; dx$. *answer*: 768

If the boundaries of integration were given as functions rather than numbers, we would have inserted them in the same way we did for double integrals.

What if we aren't given boundaries of integration, but instead have to figure them out for ourselves?

a) Find an equation $z = F_2(x, y)$ that represents the upper surface and an equation $z = F_1(x, y)$ that represents the lower surface of D.
b) Draw a two-dimensional sketch of the projection of D onto the xy-plane and determine the appropriate boundaries for integration with respect to x and y, either

$$g_1(x) \le y \le g_2(x),\ \ a \le x \le b \quad \text{or} \quad h_1(y) \le x \le h_2(y),\ \ c \le y \le d\,.$$

Example B: Evaluate $\iiint\limits_D z\,dV$ where D is the wedge in the first octant cut from the cylindrical solid $y^2 + z^2 \le 1$ by the planes $y = x$ and $x = 0$. *answer*: $\frac{1}{8}$

Editorial note: I'm doing simple examples to ensure that I have time in Lecture to finish them. The practice exercises in the text are sometimes a little more complicated, and some of the integrations will take a little more time.

In the special case where $f(x, y, z) = 1$, we get $\iiint\limits_D f\ dV = \iiint\limits_D 1\ dV =$ volume of the solid region D.

Example C: Find the volume V of the solid region D within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$. *answer*: 36π

For the fun of it (and/or in case you need a quick review for homework exercises where it isn't so straightforward), evaluating $\int_{-3}^{3} 8\sqrt{9 - x^2}\ dx$ using trig substitution is worked out below.

$$x = 3\sin\theta, \quad dx = 3\cos\theta\ d\theta$$

$$x = -3 \quad \Rightarrow \quad -1 = \sin\theta \quad \Rightarrow \quad \theta = -\tfrac{\pi}{2}$$

$$x = 3 \quad \Rightarrow \quad 1 = \sin\theta \quad \Rightarrow \quad \theta = \tfrac{\pi}{2}$$

$$\int_{-3}^{3} 8\sqrt{9 - x^2}\ dx = \int_{-\pi/2}^{\pi/2} 8\sqrt{9 - 9\sin^2\theta}\,(3\cos\theta)\ d\theta = \int_{-\pi/2}^{\pi/2} 72\cos^2\theta\ d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 72\left(\frac{1}{2}\right)(1 + \cos 2\theta)\ d\theta = \left[36(\theta + \sin 2\theta)\right]_{-\pi/2}^{\pi/2}$$

$$= 36\left(\frac{\pi}{2}\right) + \sin\pi - 36\left(-\frac{\pi}{2}\right) + \sin(-\pi) = 36\pi$$

Evaluating $\int_{-3}^{3} 2x\sqrt{9 - x^2}\ dx$ involves a straight substitution $u = 9 - x^2$. I'll let you do that one yourself for practice.

Example D: Find the volume V of the solid region D enclosed between the paraboloids $z=5x^2+5y^2$ and $z=6-7x^2-y^2$. *answer*: $\frac{3\pi}{\sqrt{2}}$

We probably won't have time to go through this one in Lecture, so here's the outline.

The surface $z=6-7x^2-y^2$ opens down from its maximum. The surface $z=5x^2+5y^2$ opens up from its minimum, so the upper surface of D is given by $F_2(x,\ y)=6-7x^2-y^2$ and the lower surface of D is given by $F_1(x,\ y)=5x^2+5y^2$.

The projection of D onto the xy-plane would be found by finding the intersection of the two paraboloids. Solve $5x^2+5y^2=6-7x^2-y^2$ to get $2x^2+y^2=1$.

For yourself, for practice, solve the iterated integral

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}}\int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}}\int_{5x^2+5y^2}^{6-7x^2-y^2} dz\ dy\ dx.$$