Calculus 241, section 14.5 Triple Integrals in Cylindrical Coordinates

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Cylindrical coordinates are, IMHO, simple and straightforward. Given a function $f(x, y, z)$, while the z-coordinate remains as it is, the x and y coordinates are converted to polar coordinates using the same formulas as in section 14.2: $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$. In addition, for some of the homework exercises, it may be useful to know that $\tan\theta = \frac{y}{x}$..

Your text gives a few very useful conversions, which (for the most part) can either be derived or which make sense conceptually.

cylinder	$x^2 + y^2 = a^2 \quad \Rightarrow \quad r = a$
sphere	$x^2 + y^2 + z^2 = a^2 \quad \Rightarrow \quad r^2 + z^2 = a^2$
double circular cone	$x^2 + y^2 = a^2 z^2 \quad \Rightarrow \quad r = az$ or $z = r\cot\phi_0$
circular paraboloid	$x^2 + y^2 = az \quad \Rightarrow \quad r^2 = az$

Among other things, these will help with practice exercises 1 through 8, converting and sketching graphs.

Theorem 14.10. "Let D be the solid region between the graphs of F_1 and F_2 on R, where R is the plane region between the polar graphs of h_1 and h_2 on $[\alpha, \beta]$, with $0 \le \beta - \alpha \le 2\pi$ and $0 \le h_1(\theta) \le h_2(\theta)$ for $\alpha \le \theta \le \beta$. If f is continuous on D, then $\iiint_D f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{F_1(r\cos\theta,\, r\sin\theta)}^{F_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta) * r\, dz\, dr\, d\theta$."

The process for establishing the correct boundaries of integration is much the same as in section 14.4.

a) Find an equation $z = F_2(r, \theta)$ that represents the upper surface and an equation $z = F_1(r, \theta)$ that represents the lower surface of D.
b) Draw a two-dimensional sketch of the projection of D onto the xy-plane and determine the appropriate boundaries for integration with respect to r and θ, $h_1(\theta) \le r \le h_2(\theta)$, $\alpha \le \theta \le \beta$.

Example A: Use cylindrical coordinates to evaluate $\iiint_D x^2\, dV$ where D is the solid region bounded by the paraboloid $z = 9 - x^2 - y^2$, the cylinder $x^2 + y^2 = 9$, and the plane $z = 0$. *answer*: $\frac{243\pi}{4}$

Example B: Find the volume V of the solid region D bounded above by the paraboloid $z = x^2 + y^2$, the cylinder $x^2 + (y-1)^2 = 1$, and the plane $z = 0$. *answer*: $\frac{3\pi}{2}$

Example C: Find the volume V of the solid region D bounded above by the paraboloid $z = 8 - x^2 - y^2$ and the plane $z = -1$. *answer*: $\frac{81\pi}{2}$

Example D: Consider a solid region D bounded by the cylinder $x^2 + y^2 = 4$, the plane $z = 3$, and the xy-plane. If the mass density at any point is equal to the distance from the point to the axis of the cylinder, find the total mass m of the region D. *answer*: 16π