Calculus 241, section 14.6 Triple Integrals in Spherical Coordinates

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While cylindrical coordinates are, IMHO, simple and straightforward, spherical coordinates seem to be more intimidating to most students. We'll try to keep it simple.

Take a point in three-dimensional space. Draw a line segment connecting the origin to that point.
Measure the angle from vertical (i.e. from the positive z-axis) and call it ϕ. $[0 \le \phi \le \pi]$
Stand at the origin facing in the direction of the x-axis, measure the positive angle you have to turn to be looking toward the point, and call it θ, the same definition as in polar and cylindrical forms. $[0 \le \theta \le 2\pi]$
Measure the distance from the origin to the point and call it ρ. $\left[\rho = \sqrt{x^2 + y^2 + z^2}\right]$

When ϕ is constant $[\phi = \alpha]$ we get a cone.
When θ is constant $[\theta = \alpha]$ we get a half-plane parallel to the z-axis.
When ρ is constant $[\rho = a]$ we get a sphere.

Our goal will be to take a function expressed as $f(x, y, z)$ or $f(r, \theta, z)$ and convert it to spherical form $f(\rho, \phi, \theta)$.

From trig triangle ratios applied in the xy-plane, we already have $x = r\cos\theta$ and $y = r\sin\theta$.
Using the same trig triangle ratios applied in a plane parallel to the z-axis, we get $r = \rho\sin\phi$ and $z = \rho\cos\phi$.

We'll use the following to accomplish our goal of converting to spherical coordinates.

$$x = r\cos\theta \quad = \rho\sin\phi\cos\theta$$
$$y = r\sin\theta \quad = \rho\sin\phi\sin\theta$$
$$x^2 + y^2 = r^2 \quad = \rho^2\sin^2\phi$$
$$z = \rho\cos\phi$$

Theorem 14.11. "Let α and β be real numbers with $\alpha \le \beta \le \alpha + 2\pi$. Let h_1, h_2, F_1 and F_2 be continuous functions with $0 \le h_1 \le h_2 \le \pi$ and $0 \le F_1 \le F_2$. Let D be the solid region consisting of all points in space whose spherical coordinates (ρ, ϕ, θ) satisfy $\alpha \le \theta \le \beta, \quad h_1(\theta) \le \phi \le h_2(\theta), \quad F_1(\phi, \theta) \le \rho \le F_2(\phi, \theta)$. If f is continuous on D, then

$$\iiint_D f(x, y, z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{F_1(\phi,\theta)}^{F_2(\phi,\theta)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) * \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta."$$

The process for establishing the correct boundaries of integration uses upper surface, lower surface and projection of D onto the xy-plane, and is illustrated better by example than by words.

Example A: Use spherical coordinates to evaluate $\iiint_D z^2 \sqrt{x^2 + y^2 + z^2}\, dV$ where D is the solid region bounded by the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and the xy-plane. *answer*: $\frac{64\pi}{9}$

ExampleA (continued)

Example B: Find the volume V of the solid region D bounded above by the sphere $x^2 + y^2 + z^2 = 16$ and below by the cone $z = \sqrt{x^2 + y^2}$. *answer*: $\frac{64\pi}{3}\left(2-\sqrt{2}\right)$

Example B revisited: Find the volume V of the solid region D between the spheres $x^2 + y^2 + z^2 = 16$ and $x^2 + y^2 + z^2 = 9$, and bounded on the sides by the cone $z = \sqrt{x^2 + y^2}$. *answer*: $\frac{37\pi}{3}\left(2-\sqrt{2}\right)$