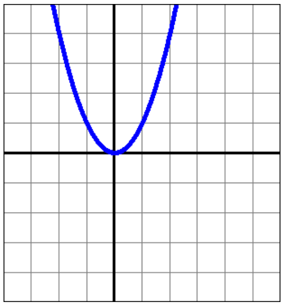
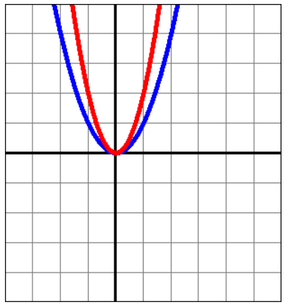
# Calculus 241, section 14.8 Change of Variables in Multiple Integrals

notes by Tim Pilachowski

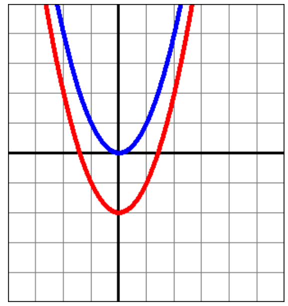
Think back to Algebra/PreCalculus. Specifically, do you remember graphing functions, and using shifts and translations?



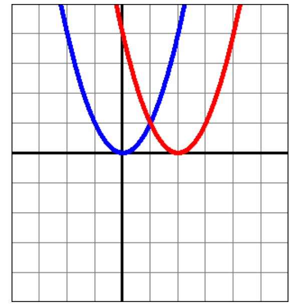
$f(x) = x^2$



$f(x) = 2x^2$



$f(x) = x^2 - 2$



$f(x) = (x-2)^2$

How about composition of functions?

$f(x) = x^2 \qquad f(2x) = (2x)^2 = 4x^2 \qquad f(3t+10) = (3t+10)^2 = 9t^2 + 60t + 100$

In Calculus I, you saw integration by substitution and the change of limits rule.

$\int_0^1 2x\sqrt{x^2-2}\,dx = \int_{-2}^{-1} u^{1/2}\,du$ where $u = x^2 - 2$

More recently, we've transformed an integration "$dz\,dy\,dx$" to an integration "$dz\,dr\,d\theta$" or "$d\rho\,d\phi\,d\,\theta$".

Each of these is an example of "change of variables", an algebraic rule that transforms one variable or function into another which either more useful or more informative. The latter two are specific applications of a more general process which we explore in today's Lecture.

Given a **transformation** $T$ which changes an integration "$dx\,dy$" to an integration "$du\,dv$", our first step is to define the **Jacobian** of $T$.

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

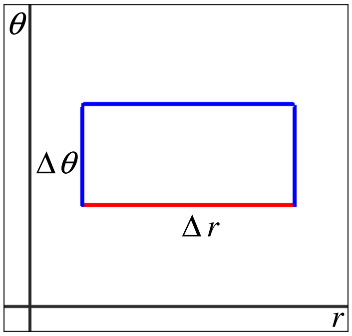
In the first few practice exercises you'll be finding a few Jacobians, and we'll be finding some in the examples below. [Good news! For this class you won't be asked to find any third-order Jacobians.]

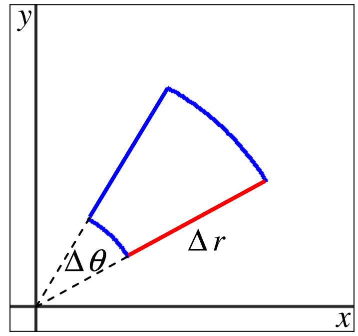
Our second step is Theorem 14.14.

"Suppose $S$ and $R$ are sets in the $uv$ and $xy$ planes respectively, each with a boundary that is a piecewise smooth curve. Let $T$ be a transformation from $S$ to $R$, defined by $x = g_1(u, v)$ and $y = g_2(u, v)$ where $g_1$ and $g_2$ have continuous partial derivatives on $S$. Suppose also that each point $(x, y)$ in $R$ is the image of a unique point $(u, v)$ in $S$ and that $\frac{\partial(x, y)}{\partial(u, v)} \neq 0$ throughout $S$ except possibly at finitely many points. Finally, assume that $f$ is continuous and integrable on $R$. Then

$$\iint_R f(x, y)\,dA = \iint_S f(g_1(u, v), g_2(u, v)) * \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dA.\text{"}$$

Here's a quick, trimmed-down version of the text's explanation of why we need the Jacobian (in general) using rectangular-to-polar transformation as the model.




"One can think of $r$ as the 'magnification factor' by which the area of a sufficiently small region in the $r\theta$ plane must be multiplied to obtain the area of an associated region in the $xy$ plane."

Given the transformation $x = g(u)$, the Jacobian equals $g'(u)$. [integration by substitution]
Given the transformation $x = r\cos\theta,\ \ y = r\sin\theta$, the Jacobian equals $r$.

Given the transformation $x = \rho\sin\phi\cos\theta,\ \ y = \rho\sin\phi\sin\theta,\ \ z = \rho\cos\phi$, the Jacobian equals $\rho^2\sin\phi$.

Example A: Evaluate $\displaystyle\iint_R \frac{x-y}{x+y}\,dA$ where $R$ is the region bounded by the lines $x - y = 0$, $x - y = 1$, $x + y = 1$ and $x + y = 3$, using the transformations $x = \dfrac{1}{2}(u+v)$ and $y = \dfrac{1}{2}(u-v)$. *answer*: $\frac{1}{4}\ln(3)$.

Why do a transformation?
In Example A, integration over the parallelogram region $R$ would have required setting up three separate integrals. After the transformation, we had only one. (See also Example B below.)

Another (very useful) reason for a transformation is to change an integration we cannot do into one we can. (See Example C below.)

What do we do when no transformation formulas are given?
In Example A, we would have noted $x + y$ and $x - y$ in the function to be integrated and (particularly) in each of the boundaries.
It was a short step to $u = x + y$ and $v = x - y$, which would provide integer boundaries of integration in the transformation.
In other words, we would have chosen a transformation that maps a rectangle $S$ in the $uv$-plane into the parallelogram $R$ in the $xy$-plane, then solved the system of equations in $x$ and $y$ to get the transformations in terms of $u$ and $v$.

The text, on p. 963, has a step-by-step description of the process of transformation.

The text does Examples involving rectangles, parallelograms, triangles, hyperbolas and ellipses. Look over those to get a feel for how each of the transformations was chosen. I'm going to use hyperbolas and triangles.

Example B: Evaluate $\iint\limits_R e^{xy}\, dA$ where $R$ is the region in the first quadrant of the $xy$-plane bounded by the lines $y = \frac{1}{2}x$ and $y = x$, and the hyperbolas $y = \frac{1}{x}$ and $y = \frac{2}{x}$. *answer*: $\frac{1}{2}\left(e^2 - e\right)\ln(2)$.
Why do a transformation?
We want the boundaries that are curves in the $xy$-plane to become constants in the $uv$-plane.

Example C: Use a transformation to integrate $\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2 \, dy\, dx$. *answer*: $\frac{2}{9}$

Why do a transformation? We don't have a method to evaluate the integral as it is written.