Math 241 Chapter 14 **Dr. Justin O. Wyss-Gallifent**

§14.1 Double Integrals

1. These can be defined via a Riemann Sum method like in Calculus I but the net result is: We can define the *double integral of $f(x,y)$ over R*, denoted $\iint_R f(x,y)\ dA$ to be the signed volume under the graph of $f(x,y)$ within the region R. The question is how to evaluate these things. First...

2. Defn: An *iterated integral* is a nested integral. An inner integral may have limits of integration which include variables further out. We evaluate these by working from the inside out, making sure we integrate with respect to the correct variable each time.

3. Now then, onto evaluation of $\iint_R f(x,y)\ dA$.

   (a) R is *vertically simple* if R may be described as between the two functions $y = bot(x)$ and $y = top(x)$ on the interval $a \le x \le b$. In this case $\iint\limits_R f(x,y)\ dA = \int_a^b \int_{bot}^{top} f(x,y)\ dy\ dx$.

   (b) R is *horizontally simple* if R may be described as between the two functions $y = left(x)$ and $y = right(x)$ on the interval $c \le y \le d$. In this case $\iint\limits_R f(x,y)\ dA = \int_c^d \int_{left}^{right} f(x,y)\ dx\ dy$.

   Consider in both cases that if either the top, bottom, left or right function ever changes then you will need more than one integral.

4. We can reparametrize (HS to VS or VS to HS) to do an impossible integral like $\int_0^1 \int_x^1 e^{(y^2)}\ dy\ dx$.

§14.2 Double Integrals in Polar Coordinates

1. Reminder about how polar coordinates work. Shapes we'll see a lot include things like $r = 2$, $r = 3\cos\theta$, $r = 2\sin\theta$, $r = 1 + \cos\theta$ as well as vertical and horizontal lines which need to be converted. Don't forget $x = r\cos\theta$, $y = r\sin\theta$ and $x^2 + y^2 = r^2$.

2. We describe a region in polar coordinates from the point of view of a person who lives at the origin. There is a near function $r = near(\theta)$ and a far function $r = far(\theta)$ between two angles $\alpha \le \theta \le \beta$. In this case $\iint\limits_R f(x,y)\ dA = \int_\alpha^\beta \int_{near}^{far} f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta$. We'll see later where that extra r comes from. It might help to remember it's the "Jacobian r".

3. We can reparametrize (to polar) to do an impossible integral like $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \sin(x^2+y^2)\ dy\ dx$.

§14.4 Triple Integrals

1. Finding a volume analogy is tricky. Instead suppose D is a solid object in space and at any point $f(x,y,z)$ is the density around that point. Then we can define $\iiint\limits_D f(x,y,z)\ dV$ as the mass of D. The question is how to evaluate which all depends upon how to best describe D.

2. We have the following:

   (a) If D is the solid between the graphs of $z = low(x,y)$ and $z = high(x,y)$ above the region R in the xy-plane and if R is VS then $\iiint\limits_D f(x,y,z)\ dV = \int_a^b \int_{bot}^{top} \int_{low}^{high} f(x,y,z)\ dz\ dy\ dx$.

   (b) If D is the solid between the graphs of $z = low(x,y)$ and $z = high(x,y)$ above the region R in the xy-plane and if R is HS then $\iiint\limits_D f(x,y,z)\ dV = \int_c^d \int_{left}^{right} \int_{low}^{high} f(x,y,z)\ dz\ dx\ dy$.

§14.5 Triple Integrals in Cylindrical Coordinates

1. Cylindrical coordinates are just polar coordinates plus z. The thing to watch out for is how equations change. For example:
   (a) $r = 2$ is a cylinder, as are $r = 3\cos\theta$ and $r = 2\sin\theta$.
   (b) The sphere $x^2 + y^2 + z^2 = 9$ becomes $r^2 + z^2 = 9$.
   (c) The cone $z = \sqrt{x^2 + y^2}$ becomes $z = r$.
   (d) The plane $x = 2$ becomes $r\cos\theta = 2$ or $r = 2\sec\theta$.
2. If D is the solid between the graphs of $z = low(x, y)$ and $z = high(x, y)$ above the region R in the xy-plane and if R is polar then we have to convert low and $high$ to polar functions $z = low(r, \theta)$ and $z = high(r, \theta)$ in terms of r and/or θ and then
   $\iiint\limits_D f(x, y, z)\ dV = \int_\alpha^\beta \int_{near}^{far} \int_{low}^{high} f(r\cos\theta, r\sin\theta, z)\ r\ dz\ d\theta\ dr.$

§14.6 Triple Integrals in Spherical Coordinates

1. Describe how spherical coordinates work and make sure to mention the conversions:
   (a) $x = \rho\sin\phi\cos\theta$
   (b) $y = \rho\sin\phi\sin\theta$
   (c) $z = \rho\cos\phi$
   (d) $x^2 + y^2 + z^2 = \rho^2$
   (e) $x^2 + y^2 = \rho^2\sin^2\phi$
2. Equations can change here. For example:
   (a) $\rho = 2$ is a sphere.
   (b) The cylinder $x^2 + y^2 = 4$ becomes $\rho = 2\csc\phi$.
   (c) $\phi = \frac{\pi}{4}$ is a cone.
   (d) The plane $z = 3$ becomes $\rho = 3\sec\phi$.
3. To describe a solid in spherical we take a range $\alpha \le \theta \le \beta$ and $\gamma \le \phi \le \delta$ From the point of view of a person at the origin this describes a "window" looking out. In that window we have a near function $\rho = near(\phi, \theta)$ and a far function $\rho = far(\phi, \theta)$.
4. If D is described this way then
   $\iiint\limits_D f(x, y, z)\ dV = \int_\alpha^\beta \int_\gamma^\delta \int_{near}^{far} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ \rho^2\sin\phi\ d\rho\ d\phi\ d\theta.$
   Don't forget that $\rho^2\sin\phi$. It's the "Jacobian" again.

§14.8 Change of Variables in Double Integrals

1. A change of variables is basically a substitution. In calc II when we did a trig sub like $x = \sin u$ we had to make sure that dx got replaced by $\cos u \; du$ and we have to do the same sort of thing here.

2. Method: If we substitute $x = f(u, v)$ and $y = g(u, v)$ then three things happen:

   (a) The region R in the xy plane changes to a new region S in the uv plane.

   (b) The integrand changes since x and y get replaced.

   (c) dA gets replaced by $|Jac| \; dA$ where Jac is the Jacobian and is defined by
   $Jac = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$

3. We have three classic examples:

   (a) $\iint\limits_R xy \; dA$ for R the parallelogram bounded by $y = x + 1$, $y = x + 4$, $y = 4 - 2x$ and $y = 10 - 2x$. We first rewrite the bounds as $y - x = 1$, $y - x = 4$, $2x + y = 4$ and $2x + y = 10$ and then put $u = y - x$ and $v = 2x + y$. The new region S is the rectangle bounded by $u = 1$, $u = 4$, $v = 4$ and $v = 10$. In order to find Jac though we need to solve for x and y. We get $x = -\frac{1}{3}u + \frac{1}{3}v$ and $y = \frac{2}{3}u + \frac{1}{3}v$ and so $Jac = -\frac{1}{3}$ and then we go from there.

   (b) $\iint\limits_R x \; dA$ for R the elliptical disk $\frac{x^2}{9} + \frac{y^2}{16} \leq 4$. We substitute $u = \frac{x}{3}$ and $v = \frac{y}{4}$ and the new region S is the disk $u^2 + v^2 \leq 4$. We use $x = 3u$ and $y = 4v$ to find Jac and go from there. The catch to this is that the new region is a disk so polar makes sense but now it's polar in the uv-plane. We use $u = r\cos\theta$ and $v = r\sin\theta$.

   (c) $\iint\limits_R y \; dA$ for R the region bounded by $y = x$, $y = 3x$, $y = 1/x$ and $y = 5/x$. We first rewrite the bounds as $y/x = 1$, $y/x = 3$, $xy = 1$ and $xy = 5$ and then put $u = y/x$ and $v = xy$. Like the first example we have to solve for x and y. I like this example because Jac has a variable in it whereas the other examples have a constant Jac.

§14.9 Parametrized Surfaces

1. When we parametrized a curve we wrote it as $\bar{r}(t)$ for some values of t. Each value of t gives a vector which points from the origin to a point on the curve. The idea now is to do the same with surfaces. We'll do $\bar{r}(u,v) = x(u,v)\,\hat{\imath} + y(u,v)\,\hat{\jmath} + z(u,v)\,\hat{k}$ for some u and v. Each pair (u,v) gives a vector which points from the origin to a point on the surface. Note that the use of u and v is arbitrary and generic and mostly we'll see $\bar{r}(x,y)$, $\bar{r}(r,\theta)$, $\bar{r}(z,\theta)$ and other familiar letters.

2. The best way to see how this works is to look at a bunch of examples. Basically the surface we're interested in somehow gets used in the parametrization and any restrictions on the surface get used in the restrictions on the variable and on the choice of variables.

   (a) The plane $z = 3$ with $0 \le x \le 3$ and $0 \le y \le 4$. This is a little rectangle. We have $\bar{r}(x,y) = x\,\hat{\imath} + y\,\hat{\jmath} + 3\,\hat{k}$ with $0 \le x \le 3$ and $0 \le y \le 4$.

   (b) The piece of $z = 3$ inside $r = 3$. This is a little disk. Since polar is better we use $\bar{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + 3\,\hat{k}$ for $0 \le r \le 3$ and $0 \le \theta \le 2\pi$.

   (c) The piece of $x = -2$ inside the cylinder $y^2 + z^2 = 9$. This is almost the same as the previous but turned on its side. We use $\bar{r}(r,\theta) = -2\,\hat{\imath} + r\cos\theta\,\hat{\jmath} + r\sin\theta\,\hat{k}$ for $0 \le r \le 3$ and $0 \le \theta \le 2\pi$.

   (d) The piece of $x^2 + y^2 = 9$ between $z = 0$ and $z = 7$. We use $\bar{r}(\theta,z) = 3\cos\theta\,\hat{\imath} + 3\sin\theta\,\hat{\jmath} + z\,\hat{k}$ with $0 \le \theta \le 2\pi$ and $0 \le z \le 7$.

   (e) The piece of the sphere $x^2 + y^2 + z^2 = 16$ inside the cone $\phi = \pi/6$. We base this off spherical and use $\bar{r}(\phi,\theta) = 4\sin\phi\cos\theta\,\hat{\imath} + 4\sin\phi\sin\theta\,\hat{\jmath} + 4\cos\phi\,\hat{k}$ with $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi/6$.

   (f) The piece of the plane $x + 2y + 3z = 12$ to the right of the rectangle in the xz-plane with opposite corners $(0,0,0)$ and $(4,0,2)$. We use $\bar{r}(x,z) = x\,\hat{\imath} + \left(\frac{12-x-3z}{2}\right)\,\hat{\jmath} + z\,\hat{k}$ with $0 \le x \le 4$ and $0 \le z \le 2$.

   Later on we'll also see examples where one (but only one) of the variables can depend upon the other one much like with VS and HS regions. For now these are good though.