Calculus 241, section 15.1 Vector Fields

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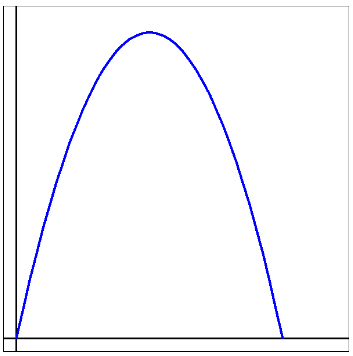
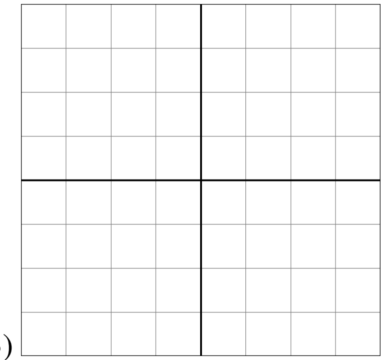
Definition 15.1: "A **vector field F** $\left[= \vec{F}\right]$ consists of two parts: a collection D of points in space, called the **domain**, and a **rule**, which assigns to each point (x, y, z) in D one and only one vector $\mathbf{F}(x, y, z)$ $\left[= \vec{F}(x, y, z)\right]$."

Examples A: 1) a projectile's trajectory, 2) flow of fluid through a pipe, 3) $\vec{F}(x, y) = 2x\vec{i} - y\vec{j}$.

1)  2)  3) 

Also take a look at the text drawings of vector fields for other visuals and for examples of how to do textbook practice homework #1 and #2.

We'll be most interested in two "derivatives" of a vector field: divergence and curl. The names originate in the study of fluid flow.

Divergence refers to the way in which fluid flows toward or away from a point. Curl refers to the rotational properties of the fluid at a point.

Example A-2 above considered only the speed (magnitude), and not the velocity (magnitude and direction). If you've ever dropped a stick or leaf into a stream, you've seen the whorls and eddies in the motion of the object as it moves with the current.

The divergence of a vector field is a formula/number (similar to directional derivative of a multivariable function). The curl of a vector field is a vector field (similar to gradient of a multivariable function).

In the formulas of this section, think of ∇ ("del") as an operator: $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$.

When applied to a function of three variables, $\nabla f(x, y, z) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = \text{grad}\, f(x, y, z)$.

Definition 15.2: Let $\mathbf{F} = \vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$. If M_x, N_x, and P_x, exist, then

$$\begin{aligned}
\text{div}\,\vec{F} &= \nabla \bullet \vec{F}(x, y, z) \\
&= \nabla \bullet \left[M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + P(x, y, z)\vec{k}\right] \\
&= \frac{\partial}{\partial x}\left[M(x, y, z)\right] + \frac{\partial}{\partial y}\left[N(x, y, z)\right] + \frac{\partial}{\partial z}\left[P(x, y, z)\right] \\
&= \frac{\partial M}{\partial x}(x, y, z) + \frac{\partial N}{\partial y}(x, y, z) + \frac{\partial P}{\partial z}(x, y, z)
\end{aligned}$$

Example B: Find the divergence of the vector field $\vec{F}(x, y) = (x^2 - y)\vec{i} + (xy - y^2)\vec{j}$.

Definitions and observations:

If $\operatorname{div} \vec{F}(x, y) = 0$, then the vector field is **divergence free** or **solenoidal**.

In physical terms, divergence refers to the way in which fluid flows toward or away from a point.
If $\operatorname{div} \vec{v}(x, y) > 0$, then the vector field is a **source**. (That is, flow is away from the point.)
If $\operatorname{div} \vec{v}(x, y) < 0$, then the vector field is a **sink**. (That is, flow is toward the point.)
If $\operatorname{div} \vec{v}(x, y) = 0$, then the vector field has neither sources nor sinks (The technical term is **incompressible**).

Definition 15.3: Let **F** $= \vec{F} = M\,\vec{i} + N\,\vec{j} + P\,\vec{k}$. If M_x, N_x, and P_x, exist, then

$$\begin{aligned}
\operatorname{curl} \vec{F} &= \nabla \times \vec{F}(x, y, z) \\
&= \nabla \times \left[M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + P(x, y, z)\vec{k} \right] \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\
&= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right)\vec{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)\vec{k}.
\end{aligned}$$

Example C: Find the curl of the vector field $\vec{F}(x, y, z) = (x^2 - y)\vec{i} + 4z\,\vec{j} + x^2\,\vec{k}$.

In physical terms, curl refers to the rotational properties of a fluid at a point. If $\operatorname{curl} \vec{F}(x, y) = \vec{0}$, then the vector field is **irrotational**.

The text notes two useful relations among gradient, divergence and curl: div $\left(\text{curl}\,\vec{F}\right)=0$, curl $\left(\text{grad}\,f\right)=\vec{0}$. The proofs need correct differentiation, and depend on the equality of the mixed partials that occur.

Example D: Find the divergence and the curl of the vector field $\vec{F}(x,\ y,\ z)=x^2 y\vec{i}+2y^3 z\vec{j}+3z\vec{k}$.

Once again, note that the divergence of a vector field is a formula/number (similar to directional derivative of a multivariable function), and the curl of a vector field is a vector field (similar to gradient of a multivariable function).

Now think back to chapter 5. A function of one variable can be "recovered" from its derivative via integration. The text does examples of recovering a function of several variables from its gradient by successive integrations. We're going to combine that process with another concept.

Theorem15.4 "Let $\vec{F}=M\vec{i}+N\vec{j}+P\vec{k}$ be a vector field. If there is a function *f* having continuous mixed partials whose gradient is $\vec{F}$, then curl $\vec{F}=\nabla\times\vec{F}(x,\ y,\ z)=\vec{0}$, that is,

$$\frac{\partial P}{\partial y}=\frac{\partial N}{\partial z},\quad \frac{\partial M}{\partial z}=\frac{\partial P}{\partial x},\quad \frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}.$$

If the domain of $\vec{F}$ is all of three-dimensional space and if the equations [above] are satisfied, then there is a function *f* such that $\vec{F}=\text{grad}\,f$."

Example E: Determine whether $\vec{F}=yze^{xyz}\,\vec{i}+\left(xze^{xyz}+2y\right)\vec{j}+xye^{xyz}\,\vec{k}$ is the gradient of some function *f*, and if so, determine *f*.

Example E continued

Example D revisited: Determine whether $\vec{F}(x,\ y,\ z) = x^2 y\,\vec{i} + 2y^3 z\,\vec{j} + 3z\,\vec{k}$ is the gradient of some function *f*, and if so, determine *f*.

Section 15.2 has a lot of stuff in it to cover, so if there's time left at the end of Lecture 15.1, we'll begin Lecture 15.2.