Calculus 241, section 15.2 Line Integrals

notes by Tim Pilachowski

In chapter 5 we introduced definite integrals: anti-derivatives over an interval $a \le x \le b$, based on a Riemann sum as the number of partitions approaches infinity.

Now we come to line integrals: anti-derivatives over a piecewise smooth curve C.

Definition 15.5: "Let f be continuous on a piecewise smooth curve C with finite length. Then the **line integral**

$$\int_C f(x, y, z)\, ds = \lim_{\|P\| \to 0} \sum_{k=1}^{9} f(x_k, y_k, z_k)\, \Delta s_k \text{."}$$

1st side note: Why "line integral" instead of "curve integral"? I have no idea.

2nd side note: If C is a closed curve, the notation $\oint_C f(x, y, z)\, ds$ is sometimes used to indicate a line integral.

As we did back in chapter 5, we won't actually calculate sums and limits. We'll be more efficient. "For the present, let us assume that C is parametrized by a smooth vector-valued function $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ on an interval $[a, b]$." Then,

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) * \left\| \frac{d\vec{r}}{dt} \right\| dt .$$

Example A. Evaluate $\int_C \left(1 + xy^2\right) ds$ where C is the line segment from (0, 0) to (1, 2).

Example A revisited. Evaluate $\int_{C_0} \left(1+xy^2\right) ds$ where C_0 is the line segment from (1, 2) to (0, 0).

Note that the results are equal. For a basic line integral, the orientation of the curve does not matter. (Orientation will matter later on, when we integrate vector fields.)

Example B. Evaluate $\int_C \left(xy+z^3\right) ds$ where C is the helix parametrized by $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$, $0 \le t \le \pi$.

Just like we have a sum rule for chapter 5 integration, we have an analogous sum rule for line integrals.
If f is continuous on a piecewise smooth curve C, composed of smooth curves $C_1, C_2, \ldots, C_n$, then

$$\int_C f(x, y, z)\,ds = \int_{C_1} f(x, y, z)\,ds + \int_{C_2} f(x, y, z)\,ds + \ldots + \int_{C_n} f(x, y, z)\,ds.$$

Example C. Evaluate $\int_C \left(x - 3y^2 + z\right) ds$ where C is composed of the two line segments from (0, 0, 0) to (1, 1, 0) and from (1, 1, 0) to (1, 1, 1).

What if we need to integrate, not a function f but rather a vector field $\vec{F}(x, y, z)$?

The text introduces this idea by using the concept of work. Consider an object moving along a smooth curve C of finite length, impelled by a force $\vec{F}$. The total amount of work done on the object can be seen as a sum of the work done to move the object along a connected series of curves, evaluated as the lengths of the curves approach 0.

Definition 15.6: "Let **F** be a continuous vector field defined on a smooth oriented curve C. Then the **line integral** of **F** over C … is defined by $\int_C \vec{F} \bullet d\vec{r} = \int_C \vec{F}(x, y, z) \bullet \vec{T}(x, y, z)\,ds$ where **T** is the tangent vector at (x, y, z) for the given orientation of C."

CAUTION: The basic line integral of Definition 15.5 does *not* require C to be oriented. The line integral of a vector field **F** *does* require C to be oriented.

From the definition of **T**, along with some algebraic manipulation, we get a workable form of Definition 15.6.

$$\int_C \vec{F} \bullet d\vec{r} = \int_a^b \vec{F}(x(t), y(t), z(t)) \bullet \frac{d\vec{r}}{dt}\,dt = \int_a^b \vec{F}(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt}\,dt$$

For the purpose of keeping this Lecture within the allotted time, Examples D, E and F will only have variables x and y. For practice homework from the text, you'll have functions of x, y, and z.

Example D. Evaluate $\int_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x, y) = (\cos x)\vec{i} + (\sin x)\vec{j}$ and C is parametrized by $\vec{r}(t) = t\vec{i} + t^2\vec{j}$, $-1 \le t \le 2$.

Example D revisited. Evaluate $\int_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x, y) = (\cos x)\vec{i} + (\sin x)\vec{j}$ and C is the line segment from $(-1, 1)$ to $(2, 4)$.

Note that the line integrals of the two versions of Example D evaluate differently for each of the two different paths.

In contrast, the text's Example 8 demonstrates a case for which the integrations of the same vector field, across two very different curves (think "paths") with a common starting point and a common terminal point, give the same result.

We'll explore the idea of "independence of path" more in section 15.3. We can relate the concept to work: If the two paths require different amounts of work, then they are not independent. If the amount of work is the same no matter which path is taken, then we have independence of path.

Since tangent vectors with opposite directions have opposite signs, it's relatively easy to show that

$$\int_{-C} \vec{F} \bullet d\vec{r} = -\int_{C} \vec{F} \bullet d\vec{r}.$$

Once again, we can get at the concept by thinking in terms of work. Going uphill we have to push. Going downhill we are moving "backward", but have to exert force to prevent the object from careening out of control.

Another way of writing the integral of a vector field in terms of its component functions M, N and P is

$$\int_C M(x, y, z)\,dx + N(x, y, z)\,dy + P(x, y, z)\,dz$$

$$= \int_a^b \left[M(x(t), y(t), z(t))\frac{dx}{dt} + N(x(t), y(t), z(t))\frac{dy}{dt} + P(x(t), y(t), z(t))\frac{dz}{dt} \right] dt.$$

Example E. Evaluate $\int_C 2xy\,dx + \left(x^2 + y^2\right)dy$ where C is the portion of the unit circle in Quadrant I of the xy-plane (standard counterclockwise orientation).

As noted above for basic line integrals, we have a sum rule.

$$\int_C \vec{F} \bullet d\vec{r} = \sum_{k=1}^{n} \int_{C_k} \vec{F} \bullet d\vec{r}$$

$$\int_C M(x, y, z)\,dx + N(x, y, z)\,dy + P(x, y, z)\,dz = \sum_{k=1}^{n} \int_{C_k} M(x, y, z)\,dx + N(x, y, z)\,dy + P(x, y, z)\,dz$$

Example F. Evaluate $\oint_C x^2 y\, dx + x\, dy$ where C is the counterclockwise path around the perimeter of the triangle with vertices (0, 0), (1, 0) and (1, 2).