Calculus 241, section 15.3 The Fundamental Theorem of Line Integrals

notes by Tim Pilachowski

In chapter 5 we had the Fundamental Theorem of Calculus.

$$\text{Given } F\text{, an antiderivative of } f,\ \int_{x_0}^{x_1} f(x)\,dx = F(x)\Big|_{x_0}^{x_1} = F(x_1) - F(x_0).$$

Now we have Theorem 15.7, the Fundamental Theorem of Line Integrals.

"Let C be an oriented curve with an initial point (x_0, y_0, z_0) and terminal point (x_1, y_1, z_1). Let f be a function of three variables that is differentiable at every point on C, and assume that grad f is continuous on C. Then

$$\int_C \text{grad } f \bullet dr = f(x_1, y_1, z_1) - f(x_0, y_0, z_0)."$$

Side note: For functions of two variables, Theorem 15.7 gives us $\int_C \text{grad } f \bullet dr = f(x_1, y_1) - f(x_0, y_0)$.

Example A: Evaluate $\int_C \vec{F} \bullet d\vec{r}$ where $\vec{F}(x, y, z) = 2xyz\,\vec{i} + x^2 z\,\vec{j} + x^2 y\,\vec{k}$ and C is parametrized by

$\vec{r}(t) = \dfrac{t}{\sqrt{t^2+1}}\vec{i} + \sin\dfrac{\pi t^3}{2}\vec{j} + \cos \pi t^3\,\vec{k}, \quad 0 \le t \le 1.$ *answer*: $-\frac{1}{2}$

1) If vector field $\vec{F}$ is the gradient of some function f, then we'll say that $\vec{F}$ is **conservative**.

2) The Fundamental Theorem of Line Integrals tells us that, as long as $\vec{F}\ (= \text{grad } f)$ and C meet the necessary requirements, the value of the integral $\int_C \vec{F} \bullet d\vec{r}$ depends only on the initial and terminal points of C. In this case, we'll say that $\int_C \vec{F} \bullet d\vec{r}$ is **independent of path**.

3) If C is a closed, oriented curve in the domain of $\vec{F}\ (= \text{grad } f)$, i.e. the initial point is the terminal point, then $\int_C \vec{F} \bullet d\vec{r} = 0$.

4) In section 15.1, we (and the text) noted that $\text{curl}(\text{grad } f) = \vec{0}$, so that when $\vec{F} = \text{grad } f$ we have $\text{curl } \vec{F} = \vec{0}$.

If the domain of $\vec{F}\ (= \text{grad } f)$ is all of three-dimensional space, then the four statements above are equivalent.

Example B. Evaluate $\int_C y\,dx + x\,dy$ where C is parametrized by $\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j},\quad 0 \le t \le 1$. *answer*: 1

First, via 15.2 methods.

Now, via 15.3 (Fundamental Theorem of Calculus).

Because this integral is independent of path, we would get the same result for any path from (0, 0) to (1, 1), whether is the parabolic path of this Example, or a linear path [$y = x$], or a cubic path [$y = x^3$], or the path determined by $\vec{r}(t) = \dfrac{2t}{t^3+1}\vec{i} + \dfrac{3t^2}{\sqrt{t}+2}\vec{j},\quad 0 \le t \le 1$, or … (You get the idea.)

15.1 Example E revisited. Evaluate $\int_C yze^{xyz}\ dx + \left(xze^{xyz} + 2y\right)dy + xye^{xyz}\ dz$ where C is any piecewise smooth curve from

(–1, 0, 1) to (1, 1, 2). *answer*: e^2