Calculus 241, section 15.4 Green's Theorem

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In section 15.3 we had the Fundamental Theorem of Line Integrals.
"Let C be an oriented curve with an initial point (x_0, y_0, z_0) and terminal point (x_1, y_1, z_1). Let f be a function of three variables that is differentiable at every point on C, and assume that grad f is continuous on C. Then

$$\int_C \text{grad } f \bullet dr = f(x_1, y_1, z_1) - f(x_0, y_0, z_0)."$$

What if, instead of evaluating an integral across a path that is a line or a curve, we want to evaluate an integral over a region in the plane? The text proves **Green's Theorem** for simple regions in the plane.

Theorem 15.8. "Let R be a simple region in the xy plane with a piecewise smooth boundary C oriented counterclockwise. Let M and N be functions of two variables having continuous partial derivatives on R. Then

$$\int_C M(x, y)\,dx + N(x, y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA."$$

Side notes: 1) Our single line integral over C has been transformed into a double integral over R.

2) Since the boundary C must, by implication, enclose the region R, the boundary C must, by further implication, be closed.

3) Orientation "counterclockwise" (in some texts called "positive") means that, as we traverse the boundary of R, we would always have the region R to our left.

4) One big advantage of Green's Theorem is that we do not need to have a parametrization of C, as we might have needed to evaluate the original line integral. Often the Green's Theorem method will be more efficient.

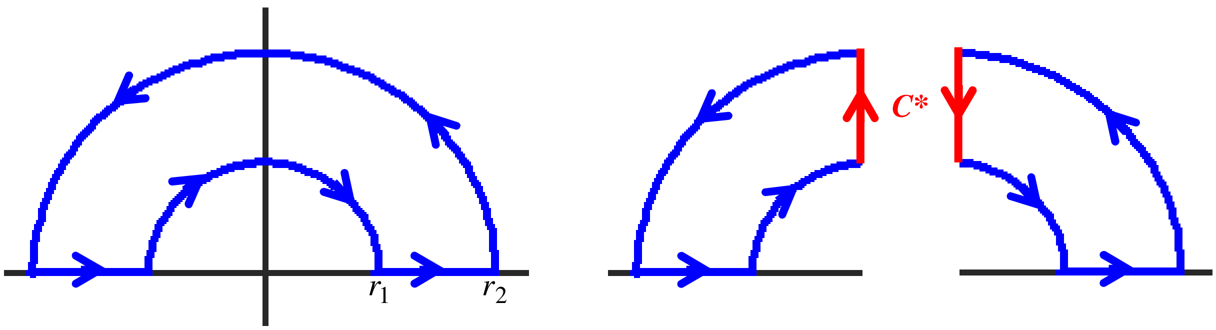
Example A: Find $\int_C M(x, y)\,dx + N(x, y)\,dy$, where $M(x, y) = x - y$, $N(x, y) = x$, and C is the unit circle, oriented counterclockwise. *answer*: 2π

15.2 Example F revisited: Evaluate $\int_C x^2 y\,dx + x\,dy$ where C is the counterclockwise path around the perimeter of the triangle with vertices ((0, 0), (1, 0) and (1, 2). *answer*: $\frac{1}{2}$

Example B: Evaluate $\int_C \left(-y^2\right)dx + xy\,dy$ where C is the counterclockwise path around the perimeter of the rectangle cut into Quadrant I by the lines $x = 1$ and $y = 2$. *answer*: 6
To do as line integral, we'd need to evaluate across curves C_1, C_2, C_3 and C_4.

Example C: Find the work done by the force field $\vec{F}(x, y) = \left(e^x - y^3\right)\vec{i} + \left(\cos y + x^3\right)\vec{j}$ on a particle that travels once around the unit circle in a counterclockwise direction. *answer*: $\frac{3\pi}{2}$

Green's Theorem holds for many regions which are not, in themselves, simple, but can be broken up into collections of simple regions. The text explores the semiannular region pictured below.



We can think of the whole region as being split up into two simple regions with a shared boundary C^*.
The traversal of each simple region counterclockwise means that we integrate across boundary C^* once going up and once going down. In effect, the two integrations zero each other out, since we showed in Lecture 15.2 that $\int_{-C} \vec{F} \bullet d\vec{r} = -\int_C \vec{F} \bullet d\vec{r}$.

The text concludes that for this non-simple region,

$$\int_C M(x, y)\,dx + N(x, y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^{\pi} \int_{r_1}^{r_2} f(r, \theta)\,dr\,d\theta.$$

You'll need to follow a similar process for text practice exercise 14.

The text then (helpfully) points out that we have three ways to evaluate a line integral of the form $\int_C \vec{F} \bullet dr$.

1) $\int_C \vec{F} \bullet dr = \int_C M\ dx + N\ dy$.
2) Provided $\vec{F}$ is the gradient of some function *f*, use the Fundamental Theorem of Line Integrals.
3) Provided that *C* is closed, use Green's Theorem.

The text also provides two alternate forms of Green's Theorem (with explanations/proofs but not Examples). These will be used primarily in working with Stokes's Theorem and the Divergence Theorem.

$$\int_C \vec{F} \bullet dr = \iint_R \text{curl } \vec{F} \bullet \vec{k}\ dA$$

$$\int_C \vec{F} \bullet \vec{n}\ dr = \iint_R \text{div } \vec{F}(x, y)\, dA \text{, where } \vec{n} \text{ is parallel to the normal vector } \vec{N} \text{ of } C.$$