Calculus 241, section 15.5 Surface Integrals

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So far, in chapter 15 we have dealt with line integrals, i.e. integrals over a curve C. Even Green's Theorem, which dealt with a region R actually focused on the boundary of R which was a curve C.

Now, beginning with section 15.5 we're moving to a higher dimension to consider integrals across a surface Σ.

Definition 15.9 "Let Σ be a smooth surface with parametrization $\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$ and assume that g is continuous on Σ. Then the **surface integral** $\iint_{\Sigma} g(x, y, z)\, dS$ is defined by

$$\iint_{\Sigma} g(x, y, z)\, dS = \iint_{R} g(x(u, v), y(u, v), z(u, v)) * \left\| \vec{r}_u(u, v) \times \vec{r}_v(u, v) \right\| dA\text{"}$$

The text also notes "… it is possible to prove that the value of the surface integral in Definition 15.9 is independent of which parametrization of $\vec{r}$ is used…the proof is technical, and we omit it."

As a way of remembering this definition, you can think of the formula as being like the change of variables formula, which was multiplied by the Jacobian.

Note that $\left\| \vec{r}_u(u, v) \times \vec{r}_v(u, v) \right\| = \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| = \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \right\|$, the norm of a 3X3 determinant.

You may have to actually evaluate some of these for some text practice exercises. We'll use some shortcuts where we can in the Examples below.

Example A. Evaluate $\iint_{\Sigma} x^2\, dS$, where Σ is the sphere $x^2 + y^2 + z^2 = 4$. *answer*: $\frac{64\pi}{3}$

Since Σ is a sphere, let's go with a spherical parametrization: $\vec{r}(\phi, \theta) = 2\sin\phi\cos\theta\,\vec{i} + 2\sin\phi\sin\theta\,\vec{j} + 2\cos\phi\,\vec{k}$.
From section 14.9 (Parametrized Surfaces) we already have

$$\left\| \vec{r}_\phi(\phi, \theta) \times \vec{r}_\theta(\phi, \theta) \right\| = \left\| \frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta} \right\| = \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix} \right\| = 4\sin\phi$$

Look familiar?

In cases where the surface Σ is a function of two variables x and y on a region R in the xy-plane, we have $\vec{r}(x, y) = x\vec{i} + y\vec{j} + f(x, y)\vec{k}$. Going back once again to section 14.9, we have

$$\left\| \vec{r}_x(x, y) \times \vec{r}_y(x, y) \right\| = \left\| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right\| = \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} \right\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}.$$

Example B. Evaluate $\iint\limits_{\Sigma} xz\, dS$, where Σ is the part of the plane $x + y + z = 1$ that lies in the first octant.

answer: $\frac{\sqrt{3}}{24}$

Example C: Evaluate $\iint\limits_{\Sigma} y^2 z^2\, dS$, where Σ is the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the planes $z = 1$ and $z = 2$. *answer*: $\frac{21\pi\sqrt{2}}{2}$

Example D: Let the surface Σ be the part of the paraboloid $z = x^2 + y^2$ that lies below the planes $z = 1$. Assume that Σ has a constant density $g(x, y, z) = 2$. Find the mass of the surface. *answer*: $\frac{\pi}{3}\left(5\sqrt{5} - 1\right)$

Two final notes: 1) The text's Example 5 evaluates an integral, for which a surface Σ is piecewise smooth, as the sum of integrals. 2) The surface area of S of Σ can be found by evaluating $\iint\limits_{\Sigma} 1\, dS$.