Calculus 241, section 15.6 Integrals over Oriented Surfaces

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In section 15.2 we looked first at line integrals of a function,

$$\int_C f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t)) * \left\| \frac{d\vec{r}}{dt} \right\| dt,$$

for which orientation was not an issue. When we turned to line integrals of a vector field,

$$\int_C \vec{F} \bullet d\vec{r} = \int_a^b \vec{F}(x(t), y(t), z(t)) \bullet \frac{d\vec{r}}{dt}\,dt = \int_a^b \vec{F}(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt}\,dt,$$

the orientation of the vector field was important.

Now, in section 15.6 we take the surface integrals of a function from section 15.5,

$$\iint_\Sigma g(x, y, z)\,dS = \iint_R g(x(u, v), y(u, v), z(u, v)) * \| \vec{r}_u(u, v) \times \vec{r}_v(u, v) \|\,dA,$$

and turn to surface integrals of a vector field, for which orientation will also be important.

For the surfaces Σ which we have considered thus far, there are two sides, which we'll think of as an "inside" and an "outside". In such a case we'll call the surface **orientable** or **two-sided**. Specifically, we'll assume that a surface Σ has a tangent plane at each of its nonboundary points, and thus that the surface has two normals pointing in opposite directions. "When Σ is the boundary of a solid region *D* in space, we customarily choose the normal to Σ that is directed outward from *D*."

Not all surfaces are orientable. Your text discusses a Möbius strip or Möbius band, which is one-sided.

You can read through the text's development of the underlying idea: a fluid of constant density δ flowing with a constant velocity $\vec{v}$ through a membrane Σ. We get a **flux integral**, $\iint_\Sigma \vec{F} \bullet \vec{n}\,dS$. [We'll refer to this type as a flux integral or we'll say "Find the flux of the vector field $\vec{F}$" even when $\vec{F}$ is not a velocity.]

Recall from section 14.9, when Σ has a parametrization

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k},$$

unit vectors which are normal to the surface are given by

$$\vec{n} = \pm\left[\frac{\vec{r}_u(u, v) \times \vec{r}_v(u, v)}{\| \vec{r}_u(u, v) \times \vec{r}_v(u, v) \|} \right] = \pm\left[\frac{\frac{\partial \vec{r}}{\partial u}(u, v) \times \frac{\partial \vec{r}}{\partial v}(u, v)}{\left\| \frac{\partial \vec{r}}{\partial u}(u, v) \times \frac{\partial \vec{r}}{\partial v}(u, v) \right\|} \right].$$

We'll choose the "+" or "–" version based on which vector has an outward orientation.

When we set up our flux integral á la section 15.5, the denominator will cancel, giving us

$$\iint_\Sigma \vec{F} \bullet \vec{n}\,dS = \pm \iint_R \vec{F}(x(u, v), y(u, v), z(u, v)) \bullet (\vec{r}_u(u, v) \times \vec{r}_v(u, v))\,dA$$

$$= \pm \iint_R \left[M(x(u, v), y(u, v), z(u, v)) * \frac{\partial(y, z)}{\partial(u, v)} + N(x(u, v), y(u, v), z(u, v)) * \frac{\partial(z, x)}{\partial(u, v)} \right.$$

$$\left. + P(x(u, v), y(u, v), z(u, v)) * \frac{\partial(x, y)}{\partial(u, v)} \right] dA$$

Looks pretty intimidating, doesn't it?
Let's put it into practice.

Example A. Find the flux of the vector field $\vec{F}(x, y, z) = z\vec{k}$ across the sphere $x^2 + y^2 + z^2 = 4$. *answer*: $\frac{32\pi}{3}$

Example B. Find the flux of the vector field $\vec{F}(x, y, z) = yz\vec{i} + x\vec{j} + z^2\vec{k}$ outward through the parabolic cylinder $y = x^2$ for $0 \le x \le 1,\ 0 \le z \le 4$. *answer*: 2

Example C. Suppose we want to find the flux of a vector field $\vec{F}(x, y, z)$ outward through the portion of the surface of the cylinder $y^2 + z^2 = 1, \quad z \geq 0$, that lies between $x = 0$ and $x = 1$. Determine the correct outward-oriented normal vector. *answer*: $n = \cos\theta \vec{i} + \sin \vec{j}$

In cases where the surface Σ is a function of two variables x and y in a region R in the xy-plane, we have $\vec{r}(x, y) = x\vec{i} + y\vec{j} + f(x, y)\vec{k}$. Going back once again to section 14.9, we have

$$\vec{r}_x(x, y) \times \vec{r}_y(x, y) = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\vec{i} - \frac{\partial f}{\partial y}\vec{j} + \vec{k}.$$

Given $\vec{F}(x, y, z) = M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + P(x, y, z)\vec{k}$, we get

$$\iint_{\Sigma} \vec{F} \bullet \vec{n}\, dS = \pm \iint_{R} \left[M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + P(x, y, z)\vec{k}\right] \bullet \left[-\frac{\partial f}{\partial x}\vec{i} - \frac{\partial f}{\partial y}\vec{j} + \vec{k}\right] dA$$

$$= \pm \iint_{R} \left[-M(x, y, f(x, y)) * \frac{\partial f}{\partial x} - N(x, y, f(x, y)) * \frac{\partial f}{\partial y} + P(x, y, f(x, y))\right] dA.$$

When "outward" is "pointing up", we'll choose the "+" version, and when "outward" is "pointing down", we'll choose the "–" version.

Example D. Let Σ be the portion of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy-plane. Find the flux of the vector field $\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$ across Σ. *answer*: 24π

Last thought: There is a sum rule for $\iint_{\Sigma} \vec{F} \bullet \vec{n}\, dS$, just like for other integrals. See the text's Example 3.