Calculus 241, section 15.7 Stokes's Theorem

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In section 15.4 we looked at Green's Theorem.

Theorem 15.8. "Let R be a simple region in the xy plane with a piecewise smooth boundary C oriented counterclockwise. Let M and N be functions of two variables having continuous partial derivatives on R. Then

$$\int_C M(x, y)\,dx + N(x, y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA."$$

Now we're going to extend from plane regions and their boundaries to three-dimensional surfaces and their boundaries with Stokes's Theorem.

Theorem 15.10. "Let Σ be an oriented surface with normal $\vec{n}$ and finite surface area. Assume that Σ is bounded by a closed, piecewise smooth curve C whose orientation is induced by Σ … Let $\vec{F}$ be a continuous vector field defined on Σ, and assume that the composite functions of $\vec{F}$ have continuous partial derivatives at each nonboundary point of Σ. Then

$$\int_C \vec{F} \bullet d\vec{r} = \iint_\Sigma \left(\text{curl } \vec{F}\right) \bullet \vec{n}\, dS.$$

If $\vec{F} = M\,\vec{i} + N\,\vec{j} + P\,\vec{k}$, then

$$\int_C M(x, y, z)\,dx + N(x, y, z)\,dy + P(x, y, z)\,dz = \iint_\Sigma \left(\text{curl } \vec{F}\right) \bullet \vec{n}\, dS."$$

Side notes: 1) Stokes's Theorem is a direct extension of the first alternate form of Green's Theorem introduced at the end of Lecture 15.4: $\int_C \vec{F} \bullet dr = \iint_R \text{curl } \vec{F} \bullet \vec{k}\, dA$.

2) Since the boundary C must, by implication, enclose the surface Σ, the boundary C must, by further implication, be closed.

3) The text states that the proof of Stokes's Theorem is beyond the scope of this course, but does give an outline of the proof for a narrow situation covered by Stokes's Theorem.

4) Once we get the integration with the curl set up, the evaluation will look like section 15.6,

$$\int_C \vec{F} \bullet d\vec{r} = \iint_\Sigma \left(\text{curl } \vec{F}\right) \bullet \vec{n}\, dS = \pm \iint_R \left[\text{curl } \vec{F}\left(x(u, v), y(u, v), z(u, v)\right)\right] \bullet \left(\vec{r}_u(u, v) \times \vec{r}_v(u, v)\right) dA.$$

Example A. Find the work performed by force field $\vec{F}(x, y, z) = x^2\,\vec{i} + 4xy^3\,\vec{j} + y^2 x\,\vec{k}$ on the particle that traverses a rectangle in the plane $z = y$ with $0 \le x \le 1$ and $0 \le y \le 3$ and with a downward orientation.

answer: -90

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Example A (continued)

The text considers a similar situation in Example 1, with the differences that the surface Σ is a triangle (so that the boundaries for y involve a function of x rather than a number value) and the induced orientation is upward.

Example B. Given the vector field $\vec{F}(x,\, y,\, z) = 2z\vec{i} + 3x\vec{j} + 5y\vec{k}$, evaluate $\int_C \vec{F} \bullet d\vec{r}$ where Σ is the portion of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane. Assume C has its counterclockwise orientation as viewed from above. *answer*: 12π