Calculus 241, section 15.8 The Divergence Theorem (Gauss's Theorem)

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At the end of section 15.4 we introduced an alternate version of Green's Theorem:

$$\int_C \vec{F} \bullet \vec{n}\, dr = \iint_R \text{div } \vec{F}(x, y)\, dA, \text{ where } \vec{n} \text{ is parallel to the normal vector } \vec{N} \text{ of } C.$$

Now we extend this idea to a **simple solid region** *D*, that is a region whose projection is simple with respect to the *xy*-plane, the *xz*-plane, and the *yz*-plane.

Theorem 15.11. "Let *D* be a simple solid whose boundary surface Σ is oriented by the normal $\vec{n}$ directed outward from *D*, and let $\vec{F}$ be a vector field whose composite functions have continuous partial derivatives on *D*. Then

$$\iint_\Sigma \vec{F} \bullet \vec{n}\, dS = \iiint_D \text{div } \vec{F}(x, y, z)\, dV."$$

Side notes: 1) The text proves the Divergence Theorem. I'll let you look at that on your own.
2) The Divergence Theorem takes an integral across a surface and transforms it into an integral over a region, usually one which is easier to evaluate.
3) Like Green's Theorem which was proven for a simple region then extended to non-simply connected regions, the Divergence Theorem holds for any solid region that can be decomposed into finitely many simple solid regions.

Example A. Find the flux of $\vec{F} = xy\,\vec{i} + yz\,\vec{j} + xz\,\vec{k}$ outward through the surface of the cube cut from the first octant by the planes planes $x = 2$, $y = 2$ and $z = 2$. *answer*: 24

Example B. Find the outward flux of $\vec{F} = x^3\,\vec{i} + y^3\,\vec{j} + z^2\,\vec{k}$ across the surface of the region enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 2$. *answer*: 279π

Example C. Find the outward flux of $\vec{F} = x^3\,\vec{i} + y^3\,\vec{j} + z^3\,\vec{k}$ across the surface of the region enclosed by the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and the plane $z = 0$. *answer*: $\frac{192\pi}{5}$

15.6 Example A revisited. Find the flux of the vector field $\vec{F}(x,\ y,\ z) = z\vec{k}$ across the sphere $x^2 + y^2 + z^2 = 4$.
answer: $\frac{32\pi}{3}$
In Lecture 15.6, this example took the equivalent of half of a page of work.