Math 241 Chapter 15 **Dr. Justin O. Wyss-Gallifent**

§15.1 Vector Fields

1. Define a vector field: Assigns a vector to each point in the plane or in 3-space. Can be visualized as loads of arrows. Can represent a force field or fluid flow - both are useful.

2. Two important definitions. Often before I do these I define $\nabla = \frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\,\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}$ so that gradient, divergence and curl all make sense with how ∇ is used.

   (a) The *divergence* $\nabla \cdot \bar{F} = M_x + N_y + P_z$ gives the net fluid flow in/out of a point (very small ball).

   (b) The *curl* $\nabla \times \bar{F}$ gives the axis of rotation of the fluid at a point.

3. For a function f we saw the gradient ∇f is a VF. In fact it's a special kind of VF. Any VF which is the gradient of a function f is *conservative* and the f is a *potential function.*
   There are two facts to note:

   (a) If $\bar{F}$ is conservative then $\nabla \times \bar{F} = \bar{0}$ and consequently if $\nabla \times \bar{F} \neq \bar{0}$ then $\bar{F}$ is not conservative. Moreover if $\nabla \times \bar{F} = \bar{0}$ and $\bar{F}$ is defined for all (x, y, z) then $\bar{F}$ is conservative.

   (b) If we have $\bar{F}$ we can tell if it's conservative by the above method and we can find the potential function too using the iterative method. Make sure to do 2-variable and 3-variable cases.

§15.2 Line Integrals (of Functions and of VFs)

1. If C is a curve and f gives the density at any point then we can define *the line integral of f over/on C*, denoted $\int_C f\ ds$, as the total mass of C. We evaluate it by parametrizing C as $\bar{r}(t)$ on $[a, b]$ and then $\int_C f\ ds = \int_a^b f(x(t), y(t), z(t))||\bar{r}'(t)||\ dt$. The result is independent of the parametrization and the orientation.
   Sample units: C in cm, f in g/cm and the result in g.

2. If C is the path of an object through a force field $\bar{F}$ then we can define *the line integral of $\bar{F}$ over/on C*, denoted $\int_C \bar{F} \cdot d\bar{r}$, as the total work done by $\bar{F}$ as it traverses C. The most basic way to evaluate it is by parametrizing C as $\bar{r}(t)$ on $[a, b]$ and then $\int_C \bar{F} \cdot d\bar{r} = \int_a^b \bar{F}(x(t), y(t), z(t)) \cdot \bar{r}'(t)\ dt$.
   Some notes about line integrals of vector fields:

   (a) The orientation (direction) of C matters. If $-C$ is the same curve in the opposite direction then $\int_{-C} \bar{F} \cdot d\bar{r} = -\int_C \bar{F} \cdot d\bar{r}$. This makes sense for work done.

   (b) The parametrization in that direction doesn't matter.

   (c) There is alternate notation for this integral. We can write $\int_C M\ dx + N\ dy + P\ dz$ which means the same as $\int_C (M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}) \cdot d\bar{r}$. Watch out for things like $\int_C M\ dx$ which looks deceivingly like a regular integral.

   Sample units: C in cm, $\bar{F}$ in $\mathrm{g} \cdot \mathrm{cm/s}$ (dynes) and the result in $\mathrm{g} \cdot \mathrm{cm}^2/\mathrm{s}^2$ (ergs).

§15.3 The Fundamental Theorem of Line Integrals

1. Thm: If $\bar{F}$ is conservative with potential f then $\int_C \bar{F} \cdot d\bar{r} = f(\text{endpoint of C}) - f(\text{startpoint of C})$.
2. Two notes:
   (a) If C is closed and $\bar{F}$ is conservative then $\int_C \bar{F} \cdot d\bar{r} = 0$.
   (b) If $\bar{F}$ is conservative we say that the integral $\int_C \bar{F} \cdot d\bar{r}$ is *independent of path* because only the start and endpoints matter, not the path taken.

§15.4 Green's Theorem

1. Thm: If C is a closed counterclockwise curve in the xy-plane which is the edge of a region R then $\int_C M\ dx + N\ dy = \iint_R N_x - M_y\ dA$.
2. Some notes:
   (a) C must be closed.
   (b) This is the same as $\int_C (M\,\hat{\imath} + N\,\hat{\jmath}) \cdot d\bar{r}$.
   (c) If C is not counterclockwise then we must negate C to make it work:
   $\int_C M\ dx + N\ dy = -\int_{-C} M\ dx + N\ dy = -\iint_R N_x - M_y\ dA$.
   (d) If R contains holes then C is all the edges (made up of pieces) and the inner holes must have clockwise orientation.
   (e) This can be sweet when $N_x - M_y$ is a constant in which case the result is a multiple of the area of R.

§15.5 Surface Integrals of Functions

1. If Σ is a surface and f gives the density at any point then we can define *the surface integral of f over/on* Σ, denoted $\iint_\Sigma f\ dS$, as the total mass of Σ. We evaluate it by parametrizing Σ as $\bar{r}(u,v)$ on the region R in the uv-plane and then $\iint_\Sigma f\ dS = \iint_R f(x(u,v), y(u,v), z(u,v)) ||\bar{r}_u \times \bar{r}_v||\ dA$.
   Sample units: Σ in cm^2, f in g/cm^2 and the result in g.
2. In this section I'll do parametrizations where one variable depends on the other. At this point we're comfortable enough (hopefully!) to understand these pretty easily.

§15.6 Surface Integrals of Vector Fields

1. Comment on oriented versus nonoriented surfaces and on fluid flow. In reality to say $\bar{F}$ is a fluid flow we really mean $\bar{F} = \delta\bar{v}$ where δ is the density at each point and $\bar{v}$ gives the velocity at each point.
2. If Σ is an oriented surface (with a sense of direction through) and $\bar{F}$ gives the fluid flow then we can define the *surface integral of $\bar{F}$ over/on* Σ, aka the *flux integral*, denoted $\iint_\Sigma \bar{F} \cdot \bar{n}\ dS$, as the total fluid flow through Σ in the direction given by the orientation. The most basic way to evaluate it is by parametrizing Σ as $\bar{r}(u,v)$ on the region R in the uv-plane and then $\iint_\Sigma \bar{F} \cdot \bar{n}\ dS = \pm \iint_R \bar{F}(x(u,v), y(u,v), z(u,v)) \cdot (\bar{r}_u \times \bar{r}_v)\ dA$ where we use $+$ if $\bar{r}_u \times \bar{r}_v$ points in the same direction as the preferred orientation and $-$ otherwise.
   Sample units: Σ in cm^2, $\bar{F}$ in g/(s $\cdot$ cm^2) and the result in g/s.
3. Important note: The use of $\bar{n}$ is to a large degree just notation and can be ignored. However If the surface is very very simple (like a horizontal plane) then we can find $\bar{n}$ directly and just do $\bar{F} \cdot \bar{n}$ first and then it becomes an integral from §15.5.

§15.7 Stokes' Theorem

1. Discuss induced orientations.
2. Thm: If Σ is a surface with oriented edge C then $\int_C \bar{F}\cdot d\bar{r} = \iint_\Sigma (\nabla\times\bar{F})\cdot\bar{n}\ dS$ where the orientation on Σ is induced from C. Again note that the left side often appears as $\int_C M\ dx + N\ dy + P\ dz$.
3. Some notes:
   (a) We'd use this when the edge is complicated but the surface is fairly easy to parametrize, a bit like Green's Theorem.
   (b) It's interesting (not heavily used by us) that this can be used when integrating $\nabla \times \bar{F}$ over some Σ_1 because we can replace Σ_1 by another surface Σ_2 provided they have the same boundary curve C via $\iint_{\Sigma_1} (\nabla \times \bar{F}) \cdot n\ dS = \int_C \bar{F} \cdot \bar{n}\ dS = \iint_{\Sigma_2} (\nabla \times \bar{F}) \cdot n\ dS$ provided we're careful about orientations.

§15.8 The Divergence Theorem (Gauss' Theorem)

1. Thm: If D is a solid object and if Σ is the boundary (outside surface) of D with outward orientation then $\iint_\Sigma \bar{F} \cdot \bar{n}\ dS = \iiint_D \nabla \cdot \bar{F}\ dV$.
2. Some notes:
   (a) Note that Σ must completely surround D.
   (b) If Σ is oriented inwards we just reverse, meaning put on a negative sign.
   (c) Watch out for shorcuts when $\nabla \cdot \bar{F}$ is a constant then the right side is just a multiple of the volume of D.