# STAT 400 – SETS AND VENN DIAGRAMS

A collection of objects is called a **set**. The individual objects in a set are called the **elements** of the set. Every set has three properties.

a. Each element of the set is unique; there can never be more than one of the same set element.
b. The members of a set are unordered; it does not matter which one is listed first and which one is listed last.
c. An element can either be a member to a set or not; there are no in-betweens.

Venn diagrams can be used to keep track of either the elements in a set or their associated probabilities.

From two given sets $A$ and $B$ we can make a new set that consists of all the elements of $A$ and all the elements of $B$. This new set is called the **union** of $A$ and $B$ and is represented by the symbol $A \cup B$. (The union symbol is *not* the letter U.) For example, let $A$ = {a, b, c, d} and let $B$ = {c, d, e}. Then $A \cup B$ = {a, b, c, d, e}. Notice that the elements c and d are in $A$ as well as $B$, but they are written only once in the list for $A \cup B$. The union of two sets is defined in symbols as follows: $A \cup B$ **=** {$x$: $x$ is in $A$ ***or*** $x$ is in $B$}. Note that this is a *non-exclusive* use of the word "or": the elements can be in $A$, or in $B$, or possibly in both. For any set $A$ it will always be true that $A \cup A = A$. Note also that the union of sets is both commutative and associative:

$$A \cup B = B \cup A \quad \text{and} \quad A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C).$$

From two given sets $A$ and $B$ we can make a new set that consists of all the elements that belong to both $A$ and $B$ at the same time. This new set is called the **intersection** of $A$ and $B$ and is represented by the symbol $A \cap B$. For example, let $A$ = {a, b, c, d} and let $B$ = {c, d, e}. Then $A \cap B$ = {c, d}. The intersection of two sets is defined in symbols as follows: $A \cap B$ **=** {$x$: $x$ is in $A$ ***and*** $x$ is in $B$}. For any set $A$ it will always be true that $A \cap A = A$. Note that the union of sets is both commutative and associative:

$$A \cap B = B \cap A \quad \text{and} \quad A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C).$$

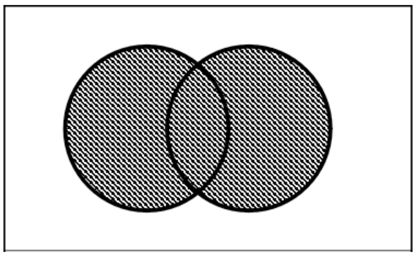
Two sets whose intersection is empty are called **disjoint**. In calculating probabilities, two events that are disjoint are called **mutually exclusive**.

Before addressing the operation of **complement**, it is necessary to define a **universal set**, containing all the individual objects being studied. For example, if the sets being studied consist of men, women, boys, and girls in a population, then the universal set is everyone in the population. In a primary school classroom, the universal set contains only positive numbers. In a typical algebra classroom, the universal set contains all real numbers, both positive and negative. For our study of probability, the universal set will be the **sample space** $S$.

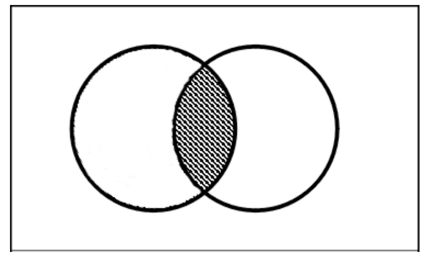
The complement of a set $A$ is the set of all elements in the universal set that do not belong to $A$ and is represented by the symbol $A'$. (I have also seen $\overline{A}$ and $A^c$ used as symbols for the complement.) It is defined in symbols as follows: $A'$ = {$x$: $x$ is not in $A$}. For example, if the universal set is the set of positive whole numbers, and $A$ is the set of all even numbers = {2, 4, 6, 8, …}, then $A'$ is the set of all odd numbers = {1, 3, 5, 7, …}. Complement literally means "that which completes", and if you combine a set with its complement, you get everything, i.e. the universe.

For any set $A$ it will always be true that $A \cup A' = S$ and $A \cap A' = \varnothing$. It will also be true that "the complement of the complement" = {$x$: not – '$x$ is not in $A$'} = {$x$: $x$ is in $A$} is the original set: symbolically $(A')' = A$. Note also that $S' = \varnothing$ and $\varnothing' = S$.

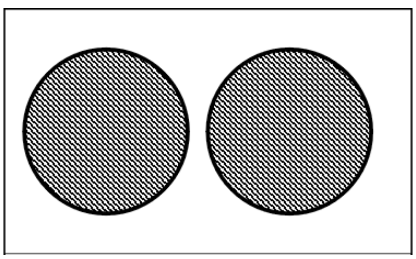
Venn diagrams provide a visual means of considering sets, even when the particular elements may not be known. In a Venn diagram a rectangle represents the universe set under consideration and circles within the rectangle represent sets within the universe. The operations above would be diagrammed as follows:



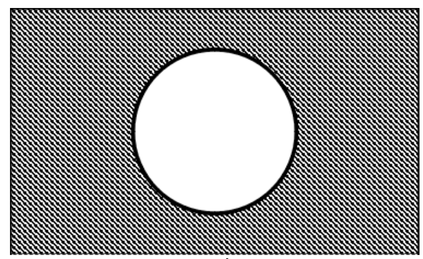
$A \cup B$



$A \cap B$

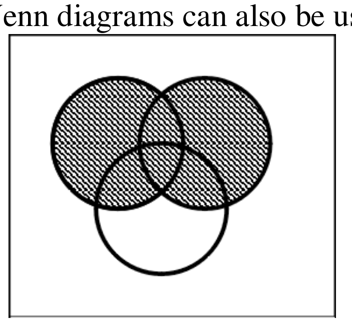


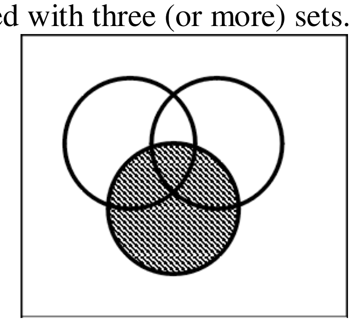
$A \cap B = \varnothing$

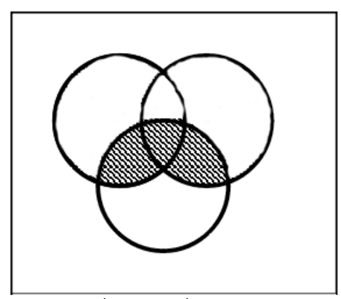


$A'$

Venn diagrams can also be used with three (or more) sets.

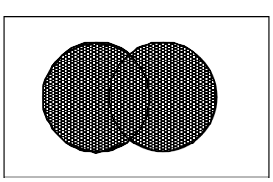






$A \cup B \quad \cap \quad C \quad = \quad (A \cup B) \cap C$

When considering the probability of a union, one must do more than add the individual probabilities. The probability of $A \cup B$ is always less than or equal to the sum of the probabilities of $A$ and $B$.



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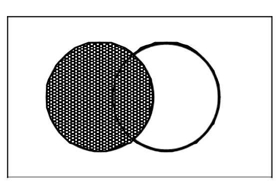

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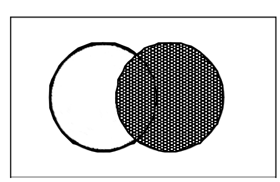

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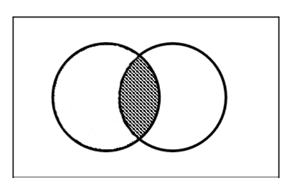


$P(A \cup B) \quad = \quad P(A) \quad + \quad P(B) \quad - \quad P(A \cap B)$

Note that $P(A \cap B)$ is included once in $P(A)$, then again in $P(B)$. The formula gives us a way to discount the duplication when calculating the probability of the union.

This formula is at least very important, and may be the most important one in this section.

Note that this formula works for mutually exclusive events, too, giving us

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - 0 = P(A) + P(B).$$

Text exercise 2.2 #22

Given: $P$(stop at first signal) = $P(L_1) = 0.4$;  $P$(stop at second signal) = $P(L_2) = 0.5$;

$P$(stop at at least one signal) = $P$(stop at first ***or*** second ***or*** both) = $P(L_1 \cup L_2) = 0.6$.

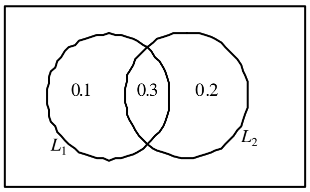
a. $P$(stop at both signals) = $P$(stop at first ***and*** second) = $P(L_1 \cap L_2)$ = ?.

We need the formula from above.

$$P(L_1 \cup L_2) = P(L_1) + P(L_2) - P(L_1 \cap L_2)$$

$$0.6 = 0.4 + 0.5 - P(L_1 \cap L_2)$$

$$P(L_1 \cap L_2) = 0.4 + 0.5 - 0.6 = 0.3$$




We can use this result to create a Venn diagram.

Note that the $L_1$ circle (left crescent + football) adds up to 0.4, as it should.

Note that the $L_2$ circle (football + right crescent) adds up to 0.5, as it should.

b. $P$(stop at first ***but not*** at second signal) = $P(L_1 \cap L_2')$.

In the Venn diagram above, this is the left crescent, so the answer is 0.1.

Formulaically, $P(L_1 \cap L_2') = P(L_1) - P(L_1 \cap L_2) = 0.4 - 0.3 = 0.1$.

c. $P$(stop at exactly one signal) = $P$(stop at first ***or*** second ***but not*** both) = $P\left[(L_1 \cup L_2) \cap (L_1 \cap L_2)'\right]$

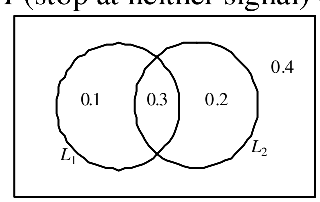
In the Venn diagram above, this is the left crescent and the right crescent, so the answer is 0.1 + 0.2 = 0.3.

Formulaically,

$$P\left[(L_1 \cup L_2) \cap (L_1 \cap L_2)'\right] = \left[P(L_1) + P(L_2) - P(L_1 \cap L_2)\right] - P(L_1 \cap L_2) = 0.4 + 0.5 - 0.3 - 0.3 = 0.3.$$

d. (my additional question)

$P$(stop at neither signal) = $P$(stop at zero signals) = $P$(not stop at first ***and*** not stop at second)




In the Venn diagram above, this is the space outside both circles, so the answer is 1 – 0.1 – 0.3 – 0.2 = 0.4.

Note that the sum of the four numbers is $P(S)$ which must equal 1.

Formulaically, we can calculate from the third piece of given information:

$P$(stop at neither signal) = $P$(stop at zero signals) = $P$(***not*** stop at one ***or*** two signals) =

$P$(complement of "stop at first ***or*** second or both") = $P(L_1 \cup L_2)' = 1 - P(L_1 \cup L_2) = 1 - 0.6 = 0.4$.

**Aha!** This is one of *De Morgans Laws*: $L_1' \cap L_2' = (L_1 \cup L_2)'$.