Stat 400, section 2.2 Axioms, Interpretations and Properties of Probability

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In section 2.1, we constructed sample spaces by asking, "What could happen?"

Now, in section 2.2, we begin asking and answering the question, "How likely are the sample points and events from our sample space?"

Concept review: A **sample space** S contains *all possible* outcomes for an experiment.
The outcomes in a sample space must be **mutually exclusive**.
An **event** (designated with a capital letter A, B, C, etc.) is a subset of the sample space, and will incorporate one or more of the outcomes.

Example A: You toss two coins. The sample space is S = { }.
a) What are the probabilities for the simple events described in the sample space?

A **probability model** assigns probabilities to all the events in a sample space. In essence, we will define the probability of an event as "the proportion of times the event is expected to occur". For simple events that are equally likely to occur, we can use a "uniform probability model", as we did in Example A-a.

Formally, for an event E

The **probability** of an event $E = P(E) = \dfrac{\text{number of ways } E \text{ can happen}}{\text{number of possible outcomes}} = \dfrac{\text{number of simple events in } E}{\text{number of simple events in } S}$.

That is, if the outcomes in sample space S are equally likely, then $P(E) = \dfrac{N(E)}{N(S)} = \dfrac{N(E)}{N}$.

Example A continued: You toss two coins. The sample space is S = { HH, HT, TH, TT }, 4 simple events that are equally likely.

A = both coins are heads =

probability that both coins are heads = $P(A)$ =

B = one coin is heads and the other is tails =

P(one coin is heads and the other is tails) = $P(B)$ =

Note that $P(S)$ =

Each event (simple or compound) in a sample space will have a **probability** associated with it. These probabilities should satisfy the following axioms.

Axiom 1. For any event E, $P(E) \geq 0$.
Axiom 2. For any sample space S, $P(S) = 1$.
Axiom 3. For any infinite collection of disjoint (mutually exclusive) events E_1, E_2, E_3, …

$$P\left(E_1 \cup E_2 \cup E_3 \cup \ldots\right) = \sum_{i=1}^{\infty} P\left(E_i\right).$$

Since $P(\varnothing) = 0$, for a finite collection of events we can derive $P\left(\bigcup_{i=1}^{k} E_i\right) = P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P\left(E_i\right) = \sum_{i=1}^{k} P\left(E_i\right).$

Example B-1: You toss a standard six-sided die. The sample space is $S = \{ 1, 2, 3, 4, 5, 6 \}$, 6 simple events that are equally likely.

A = the number rolled is even =

$P(A)$ =

B = the number rolled is at least 2 =

$P(B)$ =

C = the number rolled is more than 3 =

$P(C)$ =

D = the number rolled is at most 4 =

$P(D)$ =

Note that $P(S)$ =

Example B-2: You toss two standard six-sided dice.

$$S = \{ \begin{array}{llllll} (1, 1), & (2, 1), & (3, 1), & (4, 1), & (5, 1), & (6, 1), \\ (1, 2), & (2, 2), & (3, 2), & (4, 2), & (5, 2), & (6, 2), \\ (1, 3), & (2, 3), & (3, 3), & (4, 3), & (5, 3), & (6, 3), \\ (1, 4), & (2, 4), & (3, 4), & (4, 4), & (5, 4), & (6, 4), \\ (1, 5), & (2, 5), & (3, 5), & (4, 5), & (5, 5), & (6, 5), \\ (1, 6), & (2, 6), & (3, 6), & (4, 6), & (5, 6), & (6, 6) \} \end{array}$$

A = at least one of the dice is a 1 =

$P(A)$ =

B = the sum of the two dice is 6 =

$P(B)$ =

C = the sum of the two dice is 9 =

$P(C)$ =

$A \cap B =$

Because events A and B are not mutually exclusive, Axiom 3 does not apply.

$A \cap C =$

$P(A \cap C) =$

Because events A and C are mutually exclusive, we can apply Axiom 3.

$P(A \cup C) =$

Note that $P(S)$ =

Example C: You pick a card from a standard deck of 52 cards. [4 suits: spades (S), hearts (H), diamonds (D), clubs (C); 13 cards in each suit: ace (A), king (K), queen (Q), jack (J), 10, 9, 8, 7, 6, 5, 4, 3, 2]

S = { A-S, A-H, A-D, A-C, K-S, K-H, K-D, K-C, Q-S, Q-H, Q-D, Q-C, … , 2-S, 2-H, 2-D, 2-C }. $N(S)$ = 52. The 52 outcomes are equally likely.

A = the card is an Ace =

$P(A)$ =

B = the card is a Spade =

$P(B)$ =

C = the card is the Ace of Spades =

$P(C)$ =

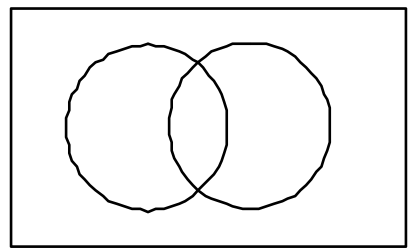
D = the card is an Ace or a Spade =

$P(D)$ =

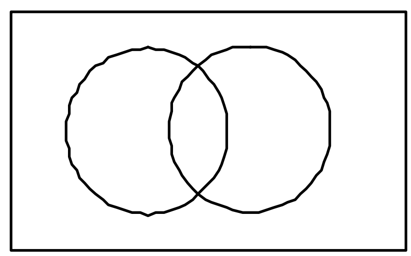
Note that $P(S)$ =

So far, in the examples we've used, it has been possible to list all of the simple events in a sample space. For larger sample spaces, this may be overwhelming, or maybe not even possible.
We are going to borrow something from set theory: Venn diagrams. Venn diagrams can be used to keep track of either the simple events in a sample space *or* their associated probabilities.

Example D: For a sample space $S = \{ s_1, s_2, s_3, s_4, s_5, s_6 \}$ all simple events are equally likely. Let $A = \{ s_1, s_2, s_3, s_4 \}$ and let $B = \{ s_3, s_4, s_5 \}$.



Venn diagram of simple outcomes



Venn diagram of probabilities

$P(A)$ = $P(B)$ =

$A \cap B =$ $P(A \cap B) =$

$N(A)$ = $N(B)$ =

$N(A \cap B) =$ $N(A \cup B) =$

$P(A \cup B) =$

the union/addition principle for probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Let $C = \{ s_6 \}$.

$A \cap C =$ $P(A \cap C) =$

$B \cap C =$ $P(B \cap C) =$

$P(A \cup C) =$

$P(B \cup C) =$

$A^c =$ $P\left(A^c\right) =$

the complement principle for probability: For any event E, $P\left(E^c\right) = 1 - P(E)$.

Example C revisited: You pick a card from a standard deck of 52 cards. [4 suits: spades (S), hearts (H), diamonds (D), clubs (C); 13 cards in each suit: ace (A), king (K), queen (Q), jack (J), 10, 9, 8, 7, 6, 5, 4, 3, 2]

A = the card is an Ace, $P(A)$ = B = the card is a Spade =, $P(B)$ = $P(A \cap B) =$

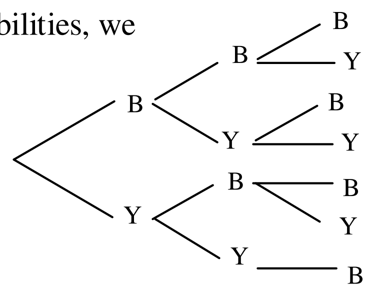
D = the card is an Ace or a Spade, $P(D)$ =

E = the card is neither an Ace nor a Spade, $P(E)$ =

Example E: Suppose that a box contains 3 blue blocks and 2 yellow blocks. You pick three blocks without replacement.

Since "blue" and "yellow" do not have uniform probabilities, we cannot simply assign each a probability of 0.5.

Experiment: Pick three blocks without replacement.



A = the first block is blue

$P(A)$ =

B = the first block is yellow

$P(B)$ =

C = given the first block is yellow, the second is blue

$P(C)$ =

From the Stat 400 page you can link to a supplement, tree diagrams and calculating probabilities, which has Example E worked out in some detail, for you to peruse at your leisure.

So far, the probabilities encountered have been theoretical. Probabilities can also be determined in **empirical** situations, through observations made about actual phenomena. The number of times a given event occurs in the long run is its **frequency**. The proportion of times a given event occurs in the long run is its **relative frequency**. In this type of situation, the probability of an event E is $P(E)$ = value to which the relative frequency stabilizes with an increasing number of trials.

Example F: A hospital records the number of days each ICU patient stays in intensive care. $S = \{ 1, 2, 3, \dots \}$. Out of 1247 ICU patients in the last 15 years, 536 stayed in ICU two weeks or less. If a patient is selected at random, what is the probability that she or he will stay in ICU two weeks or less?

Example G: Silver Springs, Florida, has a snack bar and a gift shop. The management observes 100 visitors, and counts 65 who eat in the snack bar (F), 55 who make a purchase in the gift shop (G), and 40 who do both. a) What is the probability that a visitor will not buy anything in the gift shop? b) What is the probability that a visitor will either eat in the snack bar or buy something in the gift shop? c) What is the probability that a visitor will buy something in the gift shop but not eat in the snack bar?

Example H: In 2009, households were surveyed about health insurance coverage, with the following results.

	Age 18-64	Age < 18
Public Plan	10286	10771
Private Plan	47000	15914
Uninsured	14144	1885

DATA SOURCE: CDC/NCHS, National Health Interview Survey, 1997-2009, Family Core component. Data are based on household interviews of a sample of the civilian noninstitutionalized population. Numbers given above were extrapolated from summary tables.

E_1 = individual is under 18; $P(E_1)$ =

E_2 = individual has health insurance; $P(E_2)$ =

E_3 = individual is under 18 ***<u>and</u>*** has health insurance; $P(E_3)$ =

E_4 = individual is under 18 ***<u>or</u>*** has health insurance; $P(E_4)$ =