Stat 400, section 2.3 Counting Techniques: Permutations and Combinations

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The Multiplication Principle is as follows:

Suppose a task T_1 can be peformed in n_1 ways, a task T_2 can be peformed in n_2 ways, … and finally, a task T_m can be peformed in n_m ways. Then the number of ways of performing the tasks T_1, T_2, … , T_m is given by the product n_1 times n_2 times … times n_m.

Why would this be so? Think about constructing a tree diagram. The first set of possibilities (task T_1) would have n_1 branches. From the end of *each* of these there would be n_2 branches, representing the n_2 possibilities for task T_2. From the end of *each* of these second-generation outcomes there would be n_3 branches, representing the n_3 possibilities for task T_3. And so on until we at last would come to task T_m.

Example A-1. Eddington has three blocks to play with: red, yellow, and blue. If he lays them out into a line one at a time, how many designs can he create?

Example B-1. Eddington has six blocks to play with: purple, red, orange, yellow, green and blue. If he lays them out into a line one at a time, how many designs can he create?

Example C-1. Eddington has ten blocks to play with: purple, red, orange, yellow, green, blue, white, black, gray, and brown. If he lays them out into a line one at a time, how many designs can he create?

This type of arrangement, in which the order of objects or events makes a difference, e.g. RYB ≠ RBY, is called a ***permutation***. For our text and for this class, we will assume that there is no repetition in a permutation, e.g. Edd could not and would not have two red blocks side by side because he picks a block and does not put it back.

The Multiplication Principle applied to a permutation involves what is called a ***factorial***.
For any positive integer n, "n factorial" is written and defined as

$$n! = n(n-1)(n-2)\ldots(3)(2)(1)$$

and by definition $0! = 1$.

Notes on factorials:

Example A-1 revisited. Eddington has three blocks to play with: red, yellow, and blue. If he lays two of them out into a line one at a time, how many designs can he create?

Example B-1 revisited. Eddington has six blocks to play with: purple, red, orange, yellow, green and blue. If he lays three of them out into a line one at a time, how many designs can he create?

Example C-1 revisited. Eddington has ten blocks to play with: purple, red, orange, yellow, green, blue, white, black, gray, and brown. If he lays four of them out into a line one at a time, how many designs can he create?

The process developed in Examples A-1 through C-1 can be generalized into the following formula for the number of permutations of n distinct objects taken k at a time:

$$P_{k,n} = \frac{n!}{(n-k)!}.$$

Other notations that are used and you may encounter include $P(n, k)$, ${}^{n}P_{k}$ and ${}_{n}P_{k}$.

Example D-1. Eddington has fifteen blocks to play with, each one a different color. How many ways can he pick up just one? How many ways can he line up all fifteen?

Example A-2. Eddington has three blocks to play with: red, yellow, and blue. If he picks up two of them at the same time, how many ways can he choose blocks?

Example B-2. Eddington has six blocks to play with: purple, red, orange, yellow, green and blue. If he picks up three of them at the same time, how many ways can he choose blocks?

Example C-2. Eddington has ten blocks to play with: purple, red, orange, yellow, green, blue, white, black, gray, and brown. If he picks up four of them at the same time, how many ways can he choose blocks?

Theory:

The process developed in Examples A-2 through C-2 can be generalized into the following formula for the number of combinations of n distinct objects taken k at a time:

$$\binom{n}{k} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!}.$$

Other notations that are used and you may encounter include $C(n, k)$, ${}^{n}C_{k}$ and ${}_{n}C_{k}$.

From the Stat 400 page you can link to a supplement, identifying permutations and combinations, which has a number of exercises designed to help you in identifying which situations are permutations and which are combinations.

Example D-2. Eddington has fifteen blocks to play with, each one a different color. How many combinations of 1 block are there? How many combinations of 15 blocks are there?

Note that for something like $P_{24,20}$ and $\binom{17}{18}$ the correct answer is "not possible". Why?

Now we move on to probabilities involving permutations and combinations.

Recall the probability "rules" we have so far.

For simple events that are equally likely to occur, we can use a "uniform probability model". Formally, for an event E,

$$P(E) = \frac{\text{number of ways } E \text{ can happen}}{\text{number of possible outcomes}} = \frac{\text{number of simple events in } E}{\text{number of simple events in } S} = \frac{N(E)}{N}.$$

We also have the addition principle, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, and the complement principle, $P(A') = P(S) - P(A) = 1 - P(A)$.

Example E: You toss two coins. The sample space is S = { HH, HT, TH, TT }.
Notes on probabilities and the sample space:

A = both coins are heads = { HH }, $P(A)$ =

B = exactly one coin is heads = { HT, TH }, $P(B)$ =

C = at least one coin is heads = , $P(C)$ =

We could have gotten the same results by thinking from a different perspective.
For S = two coins are tossed, $N(S)$ =

For A = both coins are heads, $N(A)$ =

For B = exactly one coin is heads, $N(B)$ =

$P(A)$ =

$P(B)$ =

$P(C)$ =

Example E extended: You toss ***<u>ten</u>*** coins and record the results.
We could list heads-tails outcomes:
S = { HHHHHHHHHH, THHHHHHHHH, HTHHHHHHHH, HHTHHHHHHH, HHHTHHHHHH, ...
TTTTTTTTTT }, but we're not going to. We don't have to. All we need to know is that, by the multiplication principle we have "2 choices for the 1st coin times 2 choices for the 2nd times...". That is,

$N(S)$ =

A = exactly three coins are heads, $P(A)$ =

B = no more than three coins are heads =

$P(B)$ =

C = at least four coins are heads =

$P(C)$ =

Other questions which ask about coins would follow the same reasoning as in the Example above. So would questions about boys and girls in families, and true-false questions on tests. In both cases, there are two choices for each "slot" to be filled in.

Example F-1: A club with twenty members is electing the three top officers: President, Vice-President, and Treasurer.

$N =$

a) What is the probability that Dickens is elected President?

b) What is the probability that Dickens is elected to one of the three officer positions?

c) What is the probability that both Dickens and his bffn Dickenson are elected as officers?

Example F-2: A club with twenty members is randomly selecting its Solstice Dance committee of 6 members.

$N =$

a) What is the probability that Dickens is selected for the committee?

b) What is the probability that both Dickens and his bffn Dickenson are selected for the committee?

c) A club has 8 male and 12 female members. If the Solstice Dance committee must have two male and four female members, what is the probability that both Dickens (male) and his bffn Dickenson (female) are selected?

$N =$

Example G: Because we can be omniscient, we know that there are 25 defective spark plugs in a production run of 1000. Quality control workers (who are not omniscient) pick ten spark plugs to test.

$N =$

a) What is the probability that all ten are defective?

b) What is the probability that at least two are defective?

Example H: A box contains 3 blue blocks and 2 yellow blocks. You pick two blocks.

$N =$

a) What is the probability of picking two blue blocks?

b) What is the probability of picking one blue and one yellow block?

c) What is the probability of picking two yellow blocks?

The rest of this Lecture's Examples involve dealing cards from a standard deck of 52.

While I am not advocating that you take up gambling, bridge and poker hands provide good examples of the interplay between permutations and combinations.

Example I: You deal a bridge hand (thirteen cards) from a standard deck of 52.

$N =$

a) What is the probability that all thirteen cards are Spades?

b) What is the probability that all thirteen cards are the same suit?

c) What is the probability that none of thirteen cards are Spades?

Example J-1: You deal four cards from a standard deck of 52.

$N =$

a) What is the probability of dealing the four Aces?

b) What is the probability of dealing four of a kind?

c) What is the probability that two of the four cards are Aces?

Example J-2: You deal a poker hand (i.e. five cards) from a standard deck of 52.

$N =$

a) What is the probability that four of the cards are Aces?

b) What is the probability of dealing a four-of-a-kind hand?

c) What is the probability that the hand is a pair of Aces (i.e. two of the cards are Aces, and there are no other matching value cards)?

d) What is the probability of dealing a pair (two-of-a-kind)?

e) What is the probability of dealing two pair (two sets of different two-of-a-kind)?

f) What is the probability of dealing a full house?

Question #43 from the text asks you to find the probabilities for getting various poker hands:

- a straight with a high card of 10, i.e. 6-7-8-9-10 with cards in any suit or suits
- a straight with any high card - Remember that an Ace can be either low card (i.e. A-2-3-4-5) or high card (i.e. 10-J-Q-K-A) in a straight. Cards can be any suit or suits. Hint: How many possible straights are there in a given suit?
- a straight flush, i.e. any high card, but all five cards must be the same suit - Hint: See Example I.