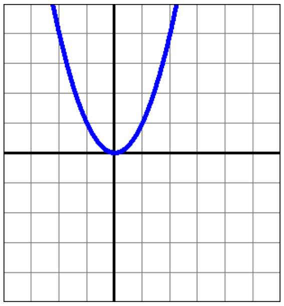
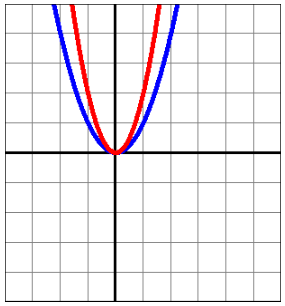
Stat 400, section 3.6b Change of Variables: Expected Value and Variance

notes prepared by Tim Pilachowski

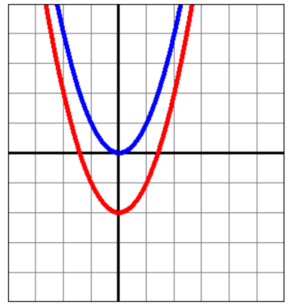
Think back to Algebra/PreCalculus. Specifically, do you remember graphing functions, and using shifts and translations?



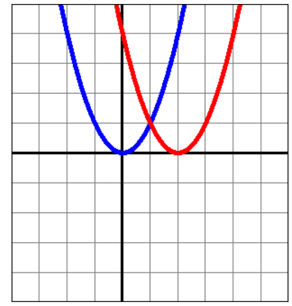
$f(x) = x^2$



$f(x) = 2x^2$



$f(x) = x^2 - 2$



$f(x) = (x-2)^2$

How about composition of functions?

$$f(x) = x^2 \qquad f(2x) = (2x)^2 = 4x^2 \qquad f(3t+10) = (3t+10)^2 = 9t^2 + 60t + 100$$

More recently, in Calculus, you saw integration by substitution and the change of limits rule.

$$\int_0^1 2x\sqrt{x^2-2}\, dx = \int_{-2}^{-1} u^{1/2}\, du \text{ where } u = x^2 - 2$$

Each of these is an example of "change of variables", an algebraic rule that transforms one variable or function into another which either more useful or more informative.

Your text covers this in a subsection of 3.3 titled "The Expected Value of a Function". Today's Lecture is a slightly more thorough exploration of this topic.

The text's Example 3.21 transforms X = number of books sold to Y = revenue from selling X books. Their Example 3.22 transforms X = number of cylinders in an engine to Y = cost to diagnose an engine with X cylinders.

Example A. Let random variable $Y = 3X + 10$, where the probability distribution for random variable X is:

x	-1	0	1
$P(X = x)$	0.2	0.3	0.5

When we transform X to Y we get:

When $X = -1$, $Y =$

When $X = 0$, $Y =$

When $X = 1$, $Y =$

The resulting probability distribution for random variable Y is:

Y			
$P(Y = y)$			

More formally, we transformed/converted the possible values of x into associated possible values for y by using an algebraic rule, i.e. function, $y = h(x) = 3x + 10$. Note also that since $h(x)$ is a one-to-one function, it is invertible and we can, at any time, transform random variable Y to random variable X using the inverse relationship $x = \dfrac{y-10}{3}$.

Consider the line graphs of the probability mass functions of random variables X and Y.

Note that, since the X-axis and Y-axis in the line graphs above are each the horizontal axis, the transformation from X to Y results in a horizontal shift on the graph. Rather than the Algebra translations referenced above, it might help you to think of Trigonometry: period calculations and phase shifts.

Example A. Let random variable $Y = 3X + 10$. Consider $E(X)$ and $E(Y)$.

$E(X) =$

$E(Y) =$

In essence, when we found $E(Y)$, we calculated

$$E(Y) = \sum_y y * p(y) = \sum_x (3x+10) * p(x) = \sum_x h(x) * p(x).$$

Something else to note: Since addition is both commutative and associative, we can rearrange the numbers in our calculation of $E(Y)$.

$E(Y) =$

Proposition: When the transformation rule $Y = h(X)$ is linear, i.e. $Y = h(X) = aX + b$, then

$$E(Y) = E(aX + b) = E(aX) + b = a\,E(X) + b$$

Using the same commutative and associative properties of addition, along with the “shortcut” version of the variance formula, we can derive $V(Y) = V(aX + b) = a^2 V(X)$.

Example A. Let random variable $Y = 3X + 10$. Find $V(X)$ and use it to calculate $V(Y)$.

In a short time, we'll come to a linear transformation which we'll use to "standardize" any set of data: $Z = \dfrac{X - \mu_X}{\sigma_X}$, which has the interesting and useful property that $\mu_Z = 0$ and $\sigma_Z = 1$.

When we get to section 5.4 and beyond, we'll extend these "change of variables for a single random variable" formulas to cases where we have a linear combination of *n* random variables.

$$Y = a_1 X_1 + a_2 X_2 + \ldots + a_n X_n \quad \Rightarrow \quad E(Y) = a_1\, E(X_1) + a_2\, E(X_2) + \ldots + a_n\, E(X_n)$$

$$V(Y) = \sigma_Y^2 = a_1^2\, V(X_1) + a_2^2\, V(X_2) + \ldots + a_n^2\, V(X_n)$$

Consider the random variable $\overline{X} = \frac{1}{n}(X_1 + X_2 + \ldots + X_n)$.

Example B. If the transformation rule is not linear, it may not be invertible. Consider the change of variable rule $Y = X^2$.

x	-1	0	1
$P(X = x)$	0.2	0.3	0.5

When $X = -1$, $Y =$

When $X = 0$, $Y =$

When $X = 1$, $Y =$

The resulting probability distribution for random variable *Y* is:

Y		
$P(Y = y)$		

Compare the line graphs of the probability mass functions of random variables *X* and *Y*.