

Stat 400, section 5.1 Jointly Distributed Random Variables

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Up to this point we have considered a single random variable X and associated probabilities for each value x that the random variable can take on. Now we turn to a *joint probability distribution* in which we have two random variables X and Y which each take on values, which in turn have associated probabilities.

Definition: Given two discrete random variables X and Y defined on a sample space S , the **joint probability mass function** is defined for each ordered pair of numbers (x, y) by

$$p(x, y) = P_{X,Y}(x, y) = P(X = x, Y = y).$$

As with probabilities we have encountered before, certain conditions must be met:

$$(1) 0 \leq p(x, y) \leq 1 \qquad (2) \sum_{\text{all } x} \sum_{\text{all } y} p(x, y) = 1.$$

You can think of this as moving from the usual Calculus I where $y = f(x)$, to multivariable Calculus in which $z = f(x, y)$.

Example A: In the game of Chinchillas and Caves™ (C&C™) there are two four-sided dice (triangular pyramids) which determine a player's next move. The purple die (X) has one side that has the number 0, one side with the number 1, and two sides with the number 2. The yellow die (Y) has sides numbered 0, 1, 1 and 3.

What are the possible outcomes?

$x \backslash y$	0	1	1	3
0				
1				
2				
2				

The **joint probability table** would look like this:

$x \backslash y$	0	1	3
0			
1			
2			

It makes sense that $P(X = x, Y = 1)$ is twice as likely as $P(X = x, Y = 0)$ and $P(X = x, Y = 3)$. Likewise, $P(X = 2, Y = y)$ is twice as likely as $P(X = 0, Y = y)$ and $P(X = 1, Y = y)$.

All probabilities meet condition (1) above, and the sum of all 9 probabilities = 1 [condition (2)].

a) What is the probability that the yellow die is at least 1?

We can extend this use of Axiom 3 to calculate the **marginal probability mass function of $X = p_X(x)$** and the **marginal probability mass function of $Y = p_Y(y)$** .

Why “marginal” probability? If we add margins to the right and bottom of the joint probability table above...

$x \backslash y$	0	1	3	$p_X(x) =$
0				
1				
2				
$p_Y(y) =$				

Important note: To get $p_X(x)$ we summed across the values of Y : $p_X(x) = \sum_{\text{all } y} p(x, y)$.

To get $p_Y(y)$ we summed across the values of X : $p_Y(y) = \sum_{\text{all } x} p(x, y)$.

b) In odd-numbered months, the length of a player’s move is determined by the sum of the numbers on the two dice. Develop the probability distribution table for random variable $B =$ length of move.

c) In even-numbered months on even-numbered dates, the length of a player’s move is determined by the product of the numbers on the two dice. Develop the probability distribution table for random variable $C =$ length of move.

d) In even-numbered months on odd-numbered dates (with the exception of February 29 in a leap year), the length of a player's move is determined by the maximum of the two numbers shown on the dice. Develop the probability distribution table for random variable $D = \max(x, y)$.

e) On February 29 in a leap year, the length of a player gets to choose whether her or his move will be determined by B , C , or D above. What might be a good strategy for making your choice?

Consider two continuous random variables X and Y defined on a sample space S with a **joint probability density function** defined for each ordered pair of numbers (x, y) by $f(x, y)$.

The definitions and observations above can be carried over from discrete to continuous random variables.

$$\text{discrete} \quad (1) 0 \leq p(x, y) \leq 1 \quad (2) \sum_x \sum_y p(x, y) = 1$$

$$\text{continuous} \quad (1) 0 \leq f(x, y) \quad (2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

Example B: Two continuous random variables X and Y have joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{2xy} & 1 \leq x \leq e, 1 \leq y \leq e^2 \\ 0 & \text{otherwise} \end{cases}.$$

The two necessary conditions are met: (1) $0 \leq f(x, y) \leq 1$ is fairly obvious, and (2) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$ was shown in Example C Lecture 5.0.

In a manner analogous to “area under the curve = probability for an interval” for a single continuous random variable X , we can state that “volume under the surface = probability for an xy region with area A ” for paired random variables X and Y .

a) What is the probability that neither X nor Y is more than 2? (This is analogous to the phrasing used in the text's Example 5.3.)

The definition of marginal probability density functions also can be carried over from discrete to continuous random variables.

$$\begin{array}{ll} \text{discrete} & p_X(x) = \sum_{\text{all } y} p(x, y) & p_Y(y) = \sum_{\text{all } x} p(x, y) \\ \text{continuous} & f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy & f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx. \end{array}$$

b) Find the two marginal probability density functions.

Next topic: independence of events

Recall from section 2.5: Two events A and B are independent when $P(A \cap B) = P(A) * P(B)$.

If we remember that $p(x, y) = P(X = x, Y = y)$ should be interpreted as

$$p(x, y) = P(X = x \text{ and } Y = y) = P(X = x \cap Y = y)$$

then it makes sense to state the following definition: Two discrete random variables X and Y are independent if

$$p(x, y) = p_X(x) * p_Y(y)$$

for every pair of x and y values.

The underlined portion is very important! Even if only one pair of matched values fails the test, then the two random variables are not independent.

In words: Two discrete random variables X and Y are independent if the joint probability $p(x, y)$ is always the product of the marginal probabilities.

Example A revisited: In the game of Chinchillas and CavesTM (C&CTM) there are two four-sided dice (triangular pyramids) which determine a player's next move. The purple die (X) has one side that has the number 0 and three sides that have the number 1. The yellow die (Y) has sides numbered 0, 1, 1 and 2. Are X and Y independent?

y				
x	0	1	3	$p_X(x) =$
0	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{4}$
1	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{4}$
2	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$
$p_Y(y) =$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

Example C (“appropriated” from Dr. Millson, Lectures 15 & 16): Toss a coin three times. Define X = number of heads on the first toss; possible values are 0, 1. Define Y = total number of heads in all three tosses; possible values are 0, 1, 2, 3. Are X and Y independent? (For practice, verify the probabilities given below.)

y x	0	1	2	3	$p_X(x) =$
0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	0	$\frac{1}{2}$
1	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$
$p_Y(y) =$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	

Example D: Random variables X and Y are independent and have marginal probability mass functions given in the table below. Reconstruct the rest of the joint probability distribution table.

y x	$\sqrt{2}$	e	π	$p_X(x) =$
$\frac{5}{3}$				0.6
2.5				0.4
$p_Y(y) =$	0.2	0.5	0.3	

The definitions of independence above can be carried over from discrete to continuous random variables.

discrete $p(x, y) = p_X(x) * p_Y(y)$

continuous $f(x, y) = f_X(x) * f_Y(y)$

Example B revisited: Two continuous random variables X and Y have joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{2xy} & 1 \leq x \leq e, 1 \leq y \leq e^2 \\ 0 & \text{otherwise} \end{cases} . \text{ Are } X \text{ and } Y \text{ independent?}$$

Note: Since $f_X(x)$ is necessarily a function of only x , and $f_Y(y)$ is necessarily a function of only y , a probability density function like the text’s Example 5.3 cannot represent independent variables.

$$f(x, y) = \frac{6}{5}(x + y^2) \neq g(x) * h(y)$$

On the other hand, a probability density function that can be written as product $f(x, y) = g(x) * h(y)$ does not necessarily represent independent random variables. See text Examples 5.5 and 5.7.

Example E. A storage facility has two independent security systems. The lifetime of the alarm on the gate has an exponential probability distribution with mean of 10 years. The lifetime of the motion sensor inside the compound has an exponential probability distribution with mean of 5 years. a) What is the joint probability density function? b) What is the probability that the facility will become unprotected in less than 3 years? c) What is the probability that at least one of the security systems will fail in less than 3 years?

The definitions of independence above can be extended from cases of two random variables to cases of more than two random variables.

$$\text{discrete} \quad p(x, y) = p_X(x) * p_Y(y) \Rightarrow p(x_1, x_2, \dots, x_n) = p_{X_1}(x) * p_{X_2}(x) * \dots * p_{X_n}(x)$$

$$\text{continuous} \quad f(x, y) = f_X(x) * f_Y(y) \Rightarrow f(x_1, x_2, \dots, x_n) = f_{X_1}(x) * f_{X_2}(x) * \dots * f_{X_n}(x)$$

Note that, using this definition, if the group of random variables X_1, X_2, \dots, X_n , is independent then each subset (pair, triple, quadruple, etc.) is necessarily independent.

However, the proposition doesn't always work the other direction. It is possible for groups of random variables to be independent when considered a pair at a time, or a triple at a time, without the group as a whole being independent.