Stat 400, section 7.2 Large Sample Confidence Intervals

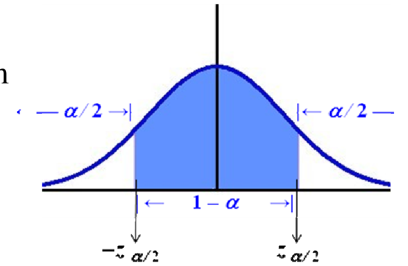
notes by Tim Pilachowski

a) Large-Sample Two-sided Confidence Interval for a Population Mean

section 7.1 redux

The point estimate for a population mean μ will be a sample mean $\bar{x}$.

a = probability of error, $1 - a$ = confidence level = probability that a random interval will capture the true value of the population parameter μ.



Values of $z_{\alpha/2}$ for common two-sided confidence levels.

$1-\alpha$	0.80	0.85	0.90	0.95	0.99
$z_{\alpha/2}$	1.28	1.44	1.645	1.96	2.58

$100(1 - \alpha)\%$ confidence interval: $\left(\bar{x} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right),\ \bar{x} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\right)$.

sample size needed to construct a $100(1 - \alpha)\%$ confidence interval with a margin of error of w: $n = \left(\frac{z_{\alpha/2}(\sigma)}{w}\right)^2$

In section 7.1, we made very important assumption: even though the parameter μ was unknown, the population's probability distribution was approximately normal and the parameter σ^2 was known and was used in the confidence interval and sample size formulas above.

Example D-background. In 1868, Carl Reinhold August Wunderlich published his definitive work on clinical thermometry. In it, he gives the normal human body temperature as 98.6º F (37º C). (He did note that "normal temperature" is a range, described variations in temperature across 24 hours, and established 100.4 F (38 C) as the first quantitative definition of fever.) A group of researchers investigated whether increased reliability of modern thermometers might challenge this "common knowledge" and/or whether human physiology may have changed over the past century.

Mackowiak, P. A., Wasserman, S. S., and Levine, M. M. (1992), "A Critical Appraisal of 98.6 Degrees F, the Upper Limit of the Normal Body Temperature, and Other Legacies of Carl Reinhold August Wunderlich," *Journal of the American Medical Association*, 268, 1578-1580., http://jama.ama-assn.org/content/268/12/1578.abstract

Example D-a. Researchers want to investigate human internal body temperature. Based on prior studies, the population probability distribution is approximately normal with a standard deviation estimated at 0.73. If they want a 95% confidence level, with a margin of error of no more than 0.2, how large should their sample be?
answer: at least 52

Now comes an important question: How can we determine a confidence interval in cases where the population probability distribution may not be normal, and where we don't know the value of σ^2?

To answer the first part, we invoke the Central Limit Theorem which implies that $\bar{X}$ has approximately a normal distribution, no matter what the shape of the distribution of X. In cases where the population probability distribution may not be normal, we will construct a confidence interval for which the level of confidence is *approximately* $100(1 - \alpha)\%$.

When the population variance σ^2 is not known (as will usually be the case) and therefore population standard deviation σ is not known, the rationale gets a little trickier.

We go back once again to the Central Limit Theorem. To answer probability questions back in section 5.4 and to construct confidence intervals in section 7.1, we worked from the linear transformation $Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$, in which only $\overline{X}$ is a random variable.

If we now turn to $Z = \frac{\overline{X} - \mu}{S / \sqrt{n}}$ (and use random variable S as well as random variable $\overline{X}$), then there is randomness in both numerator and denominator. However, if n is sufficiently large, it will ameliorate the effects of the extra variability introduced by using S. The rule of thumb for invoking the Central Limit Theorem was $n > 30$. To account for the added variability, this text proposes $n > 40$ as a rule of thumb for using the following formula for a large-sample two-sided interval with a confidence level of approximately $100(1 - \alpha)\%$:

$$\left(\overline{x} - z_{\alpha/2}\left(\frac{s}{\sqrt{n}}\right),\ \overline{x} + z_{\alpha/2}\left(\frac{s}{\sqrt{n}}\right)\right).$$

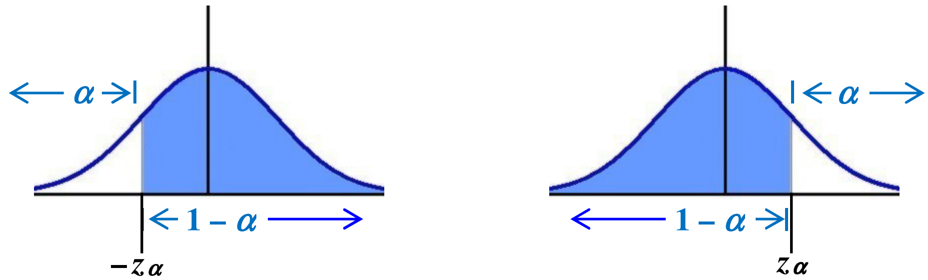
Example D-b. In their 1992 study of human internal body temperature, Mackowiak, Wasserman and Levine had a sample size of 148. Given $\overline{x} = 98.25$ and $s = 0.73$, construct a two-sided 95% confidence interval.
answer: (98.132, 98.368)

What would be the effect of increasing the level of confidence? Would the resulting confidence interval be wider or narrower?

Example D-c. In their 1992 study of human internal body temperature, Mackowiak, Wasserman and Levine had a sample size of 148. Given $\overline{x} = 98.25$ and $s = 0.73$, construct a two-sided 99% confidence interval.
answer: (98.095, 98.405)

b) Large-Sample One-sided Confidence Interval for a Population Mean

So far, the confidence intervals we've considered have been two-sided, i.e. they have both a lower and an upper bound. In some cases, a researcher will want a one-sided confidence interval, i.e. either an upper or lower bound but not both. For example, in a study about a new medication, doctors will be interested in whether the new formulation is better than the old. Or a mechanical engineer may want to investigate a lower failure rate for components.



The point estimate for a population mean μ will be a sample mean $\bar{x}$.

a = probability of error, 1 – a = confidence level = probability that a random interval will capture the true value of the population parameter μ.

100(1 – α)% two-sided confidence interval: $\bar{x} - z_{\alpha/2}\left(\frac{s}{\sqrt{n}}\right) < \mu < \bar{x} + z_{\alpha/2}\left(\frac{s}{\sqrt{n}}\right)$

upper 100(1 – α)% one-sided confidence bound: $\mu < \bar{x} + z_{\alpha}\left(\frac{s}{\sqrt{n}}\right)$

lower 100(1 – α)% one-sided confidence bound: $\mu > \bar{x} - z_{\alpha}\left(\frac{s}{\sqrt{n}}\right)$

Values of z_α for common one-sided confidence levels.

1 – α	0.80	0.85	0.90	0.95	0.99
z_α					

7.1 Example B revisited: A sample has mean = 150 and the population has known standard deviation = 22. For a random sample of size 47, find the lower bound for a one-sided 90% confidence interval. *answer*: 145.892

Example D-b revisited. In their 1992 study of human internal body temperature, Mackowiak, Wasserman and Levine had a sample size of 148. Their sample mean, $\bar{x}$ = 98.25 with s = 0.73, led them to a hypothesis that human internal body temperature is lower than the conventional 98.6º F. Does a one-sided 95% confidence interval based on their sample data include 98.6º F? *answer*: μ < 98.35; No.

c) Large-Sample Confidence Interval for a Population Proportion

When constructing a large-sample confidence interval of approximately $100(1-\alpha)\%$ for estimating population parameter μ, we used the formula

$$\bar{x} \pm z_{\alpha/2}\left(\frac{s}{\sqrt{n}}\right).$$

Expressed in generic terms, this is

(point estimate of θ) $\pm$ (critical value for z) times (estimated standard error of the estimator).

Applying this to population proportions, and using the standard error formula developed in Lecture 6.1c, we get

$$\hat{p} \pm z_{\alpha/2}\left(\sqrt{\frac{\hat{p}\hat{q}}{n}}\right).$$

6.1c Example B. A news organization polls voters and asks, "Do you intend to vote for incumbent Senator Phillip E. Buster in the upcoming election?" Pollsters record the following numbers.

	Yes	Undecided	No
Male	56	18	47
Female	82	12	35

Construct a 95% confidence interval for the proportion of voters who would favor reelecting Senator Buster.

This formula for confidence interval of a population proportion is the one that has been used in introductory statistics courses for a long time, and is often used in practice. If the sample size n is very large, and if the value of the population proportion p is close to 0.5, then $\hat{p} \pm z_{\alpha/2}\left(\sqrt{\frac{\hat{p}\hat{q}}{n}}\right)$ gives a reasonably good confidence interval.

However, it has severe limitations. For small values of n, or for values of $\hat{p}$ and $\hat{q}$ close to 0 or 1, the variability of the standard error portion of the formula increases dramatically. (See Figure 7.6 in your text.)

We'll backtrack a little, and use a process similar to that used in section 7.1.

$$P\left(-z_{\alpha/2} < \frac{\hat{p}-p}{\sqrt{p(1-p)/n}} < z_{\alpha/2}\right) \approx 1-\alpha$$

The next step is replacing the inequalities with " = " and solving for p.

$$-z_{\alpha/2} = \frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \qquad\qquad \frac{\hat{p}-p}{\sqrt{p(1-p)/n}} = z_{\alpha/2}$$

Use of the quadratic formula in an intensive algebraic process would get us to

$$p = \frac{\hat{p} + z_{\alpha/2}^2 / 2n}{1 + z_{\alpha/2}^2 / n} \pm z_{\alpha/2} \frac{\sqrt{\hat{p}\hat{q}/n + z_{\alpha/2}^2 / 4n^2}}{1 + z_{\alpha/2}^2 / n}.$$

An advantage of this more complicated version is that it addresses the deficiencies of the classic formula, and works well for almost all values of n and p, even when $np < 10$ or $nq < 10$.

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$$\frac{(\quad) + \frac{(\quad)^2}{2(\quad)}}{1 + \frac{(\quad)^2}{(\quad)}} \pm (\quad) \frac{\sqrt{\frac{(\quad)(\quad)}{(\quad)} + \frac{(\quad)^2}{4(\quad)^2}}}{1 + \frac{(\quad)^2}{(\quad)}}$$