Stat 401, section 7.1 Basic Properties of Confidence Intervals

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In the real world, researchers often don't have enough data and resources to determine actual population means and proportions (**population parameters**).

Likewise, it isn't possible to construct an entire sampling distribution to determine values of parameters. Instead, a sample will be taken and the **sample statistics** calculated.

The questions are always, "How well does this sample reflect the population? How reliable are the sample statistics as indicators of the population parameters?"

Chapter 7 uses the work of chapters 5 and 6 to begin answering these questions.

Vocabulary:
A statistic used for estimating a parameter is called a **point estimator** or **estimator**.
The standard deviation of an estimator is called its **standard error**.

The point estimate for a population mean μ will be a sample mean $\bar{x}$.
The point estimate for a population proportion p will be a sample proportion $\hat{p}$.

In this section we'll be looking at a sample as one of all of the possible samples in a **sampling distribution** and be asking, "How likely is it that our sample mean is within a given percent of the actual, but unknown, population parameter?"

Fill in the blanks.

population parameter =

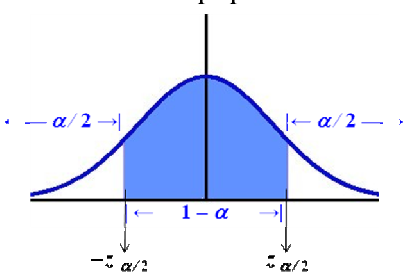
point estimate = sample mean =

standard error =

Before we can begin developing a process for determining how reliable our point estimates are, we need some more vocabulary and notation.

a = probability of error

$1 - a$ = confidence level = probability that a random interval will capture the true value of the population parameter μ.
On this picture of a sampling distribution, the confidence level is $100(1 - \alpha)\%$ surrounding μ in the middle.



Because a normal distribution is symmetric, each tail must contain $\alpha/2$.

The left and right boundaries will be the negative and positive values of z that mark the lower and upper $100(\alpha/2)\%$.

Important note: The vocabulary above and the rest of this Lecture will apply to a **two-sided confidence interval**. We'll take a look at one-sided confidence intervals in Lecture 7.2.

Example A: Determine the value of $z_{a/2}$ for a (two-sided) confidence level of 80%.   *answer*: 1.28

Using a similar process, we could find corresponding values for common two-sided confidence levels.

$1-\alpha$	0.80	0.85	0.90	0.95	0.99
$z_{\alpha/2}$					

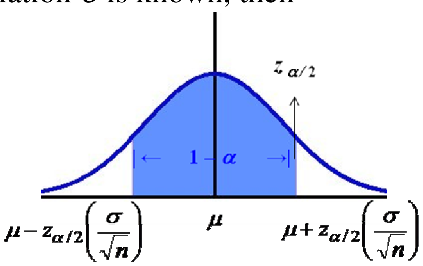
Back in section 5.4 we encountered the Central Limit Theorem, which assures us of three things.

1) As the sample size n increases, or as the number of trials n approaches infinite, the shape of a sampling distribution becomes increasingly like a normal distribution.

2) The mean of a sampling distribution = the mean of population: $E(\overline{X}) = \mu$.

3) The standard deviation of a sampling distribution = $\sigma(\overline{X}) = \dfrac{\sigma_X}{\sqrt{n}}$.

If we assume (for the moment) that a population has a normal distribution and that the population standard deviation σ is known, then



$$P\left[\ \mu - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) \ < \ \overline{X} \ < \ \mu + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\ \right] = 1-\alpha \qquad (1)$$

Now we're going to do some mathematical manipulating of the formula on the right, line (1).

$$\mu - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) \ < \ \overline{X}$$

$$\mu \ < \ \overline{X} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$$

$$\overline{X} \ < \ \mu + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$$

$$\overline{X} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) \ < \ \mu$$

$$P\left[\ \overline{X} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) \ < \ \mu \ < \ \overline{X} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\ \right] = 1-\alpha \qquad (2)$$

Lines (1) and (2) are mathematically equivalent!
Line (2) gives us the lower and upper limits of the $100(1-\alpha)\%$ *confidence interval.*

Example B – confidence interval: A sample has mean = 150 and the population has known standard deviation = 22. For a random sample of size 47, find the 90% two-sided confidence interval. *answer*: (144.721, 155.279)

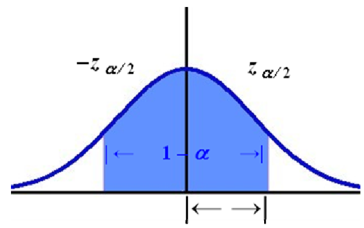
work:

interpretation

You should definitely read the text's explanation of "Interpreting a Confidence Interval".
highlights:

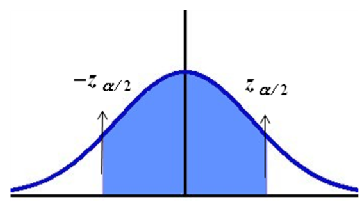
The probability involved in a confidence interval needs to be interpreted as a long-run relative frequency, which is why studies will be repeated by several researchers attempting to duplicate results. Part of hand-in homework #1 involves using Minitab to run a simulation to construct confidence intervals.

With a little further work, we can calculate the amount of a specified *margin of error*. Our question becomes, "The mean plus what amount takes us to the right-side boundary?" We can use the z-score formula for a sampling distribution to derive a formula for **margin of error**.



Important point: We're looking at a sampling distribution (which has a normal probability distribution) and are using a sample mean $\bar{x}$ (an established value) as our point estimate for the population parameter μ.

Example B – margin of error: A population has a known standard deviation = 22. For a random sample of size 47, find the 90% margin of error.
answer: ≈ 5.279



Earlier, it was noted that the Central Limit Theorem tells us, "As the sample size n increases, or as the number of trials n approaches infinite, the shape of a sampling distribution becomes increasingly like a normal distribution."

The question now becomes, "What value of n is 'large enough' for a desired confidence level"?
We'll answer by solving the margin of error formula for n, using w to represent the desired margin of error (width of the confidence interval).

Example B – sample size: A population has standard deviation $\sigma = 22$. If we wanted to be 99% confident in our results, with a margin of error of no more than 2, how large should our sample be?
answer: 806

Always take your result up to the next whole number to make sure the entire error margin is included.