Stat 401, section 8.2 z Tests for Hypotheses about a Population Mean

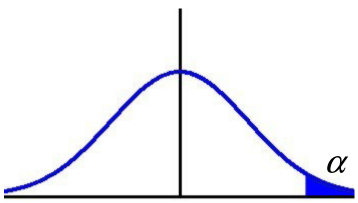
notes by Tim Pilachowski

This lecture will focus on the material in section 8.2 of the 9th edition. [This is Case I and Case II in the 8th edition of the text, along with the approach of section 8.4 of the 8th edition.]

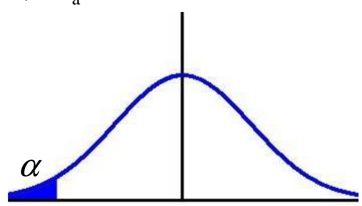
Here's the underlying concept of a hypothesis test. We'll assume we know something about a population parameter θ. If we're correct, then the sample statistic $\hat{\theta}$ that we calculate has a high probability, $100(1 - \alpha)\%$, of being close to our hypothesized θ. If, however, we get a sample statistic $\hat{\theta}$ which has a low probability of occurring, we'll question whether our initial assumption about population parameter θ was correct. (Think "proof by contradiction".)

Here are the steps as outlined in the text.

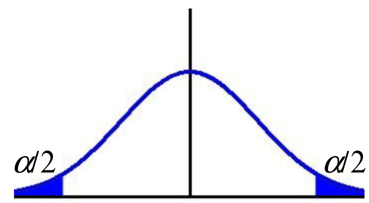
1. Identify the population parameter under scrutiny and describe it in the context of the described situation.
2. Determine the null value and state the null hypothesis, H_0.
3. State the appropriate alternate hypothesis, H_a.



$H_0: \theta \le \theta_0$
$H_a: \theta > \theta_0$



$H_0: \theta \ge \theta_0$
$H_a: \theta < \theta_0$



$H_0: \theta = \theta_0$
$H_a: \theta \ne \theta_0$

4. Give the formula for the computed value of the test statistic.
5. Compute any necessary sample statistics, and use them to compute the value of the test statistic.
6. Determine the P-value: $p = 2\left(1 - \Phi\left(|z|\right)\right)$ for a two-tailed test, $p = 1 - \Phi\left(|z|\right) = \Phi\left(-|z|\right)$ for a one-tailed test.
7. If $p \le \alpha$, reject H_0. Otherwise, fail to reject H_0. State the conclusion in the context of the problem.

If the population is normal with a known standard deviation, or if n is large enough (rule of thumb, $n > 40$), then the hypothesis test is considered a large-sample test.

Example A. To be useful, ball bearings need to have a constant mean diameter of 0.50 cm; those much larger or smaller can wreak havoc. From past experience, the diameter of the ball bearings produced has a normal distribution with a standard deviation $\sigma = 0.04$. A sample of 50 has a mean diameter of 0.51 cm. At a significance level of 0.02, does the data indicate that the production machinery needs to be adjusted?

Example B. In their 1992 study of human internal body temperature, Mackowiak, Wasserman and Levine had a sample size of 148, and calculated $\bar{x} = 98.25$ and $s = 0.73$. Does the data indicate that mean human internal body temperature is less than the long-used value of 98.6° F?

For a hypothesis test of means, the significance level α is the probability of a Type I error: Even if the null hypothesis is correct, there is still a probability of α that the calculated test statistic will fall into the "very unlikely" tail (the rejection region).

The calculation of β, the probability of a Type II error, is a good bit more problematic.

Recall that a Type II error is failing to reject the null hypothesis when the null hypothesis is actually false.

In other words, in Example A above, the null value 0.50 is not the actual value of the population mean μ.

But if we don't know what μ actually is, how can we calculate the probability β?

We need another possible value for the population mean, which we'll denote μ'.

Now the question can be phrased, "If the actual population mean is μ' instead of the hypothesized μ_0, what is the probability of failing to reject $\mu = \mu_0$?

Given the conditions that a population has a normal distribution with a known σ, the following formulas apply.

Given null hypothesis $H_0 : \mu = \mu_0$ and test statistic $Z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$,

for $H_a : \mu > \mu_0$, Type II error probability $\beta(\mu') = \Phi\left(z_\alpha + \dfrac{\mu_0 - \mu'}{\sigma/\sqrt{n}} \right)$,

for $H_a : \mu < \mu_0$, $\beta(\mu') = 1 - \Phi\left(-z_\alpha + \dfrac{\mu_0 - \mu'}{\sigma/\sqrt{n}} \right)$,

for $H_a : \mu \neq \mu_0$, $\beta(\mu') = \Phi\left(z_{\alpha/2} + \dfrac{\mu_0 - \mu'}{\sigma/\sqrt{n}} \right) - \Phi\left(-z_{\alpha/2} + \dfrac{\mu_0 - \mu'}{\sigma/\sqrt{n}} \right)$.

The sample size n for which a significance level α test also has $\beta(\mu') = \beta$ at the alternative value μ' is

$$n = \begin{cases} \left[\dfrac{\sigma(z_\alpha + z_\beta)}{\mu_0 - \mu'} \right]^2 & \text{for a one - tailed (upper or lower) test} \\ \left[\dfrac{\sigma(z_{\alpha/2} + z_\beta)}{\mu_0 - \mu'} \right]^2 & \text{for a two - tailed test (approximate solution)} \end{cases}$$

Example C. A car manufacturer claims that a new model gets, on the average, thirty miles per gallon. A consumer group believes the actual mileage is less than 30 mpg and tests 50 cars, obtaining a sample mean of 28.3. Prior testing gives reason to believe that the distribution of miles per gallon is normal, with a standard deviation of 2.21. a) If a significance level $\alpha = 0.10$ test is used, what is the probability of a Type II error when μ is actually 29 mpg? b) What value of n is necessary to ensure that $\beta(29) = 0.10$ when $\alpha = 0.10$?